Application of Kalman Filters to tracking in a magnetic field

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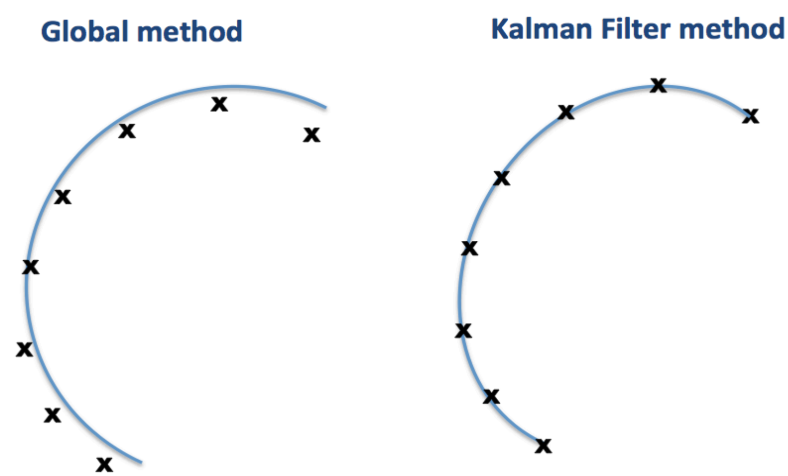
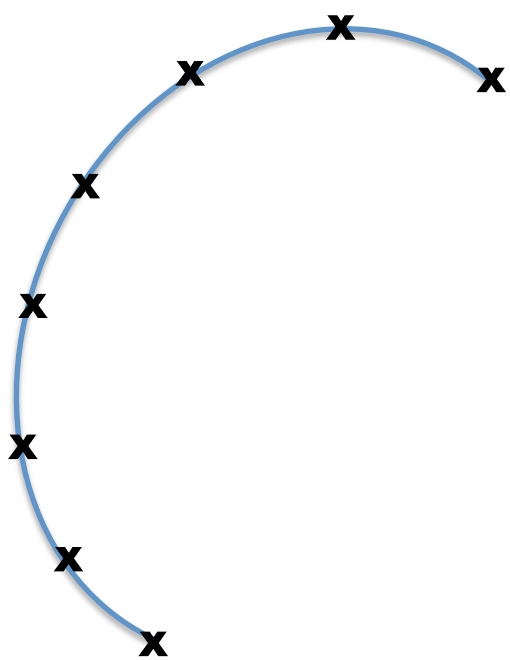
Charged particle trajectory in a approximately Uniform Magnetic Field

- In approximately constant B-field, a charged particle follows a helical path
  - standard parametrization (d_ρ,ϕ^0,q/p,dz,tanλ)
  - if there is no energy loss and multiple scattering
  - Global fitting method → simple χ^2 minimization assuming a perfect helix
- For low momentum particles ionization energy loss can be significant causing a distortion in the helical trajectory



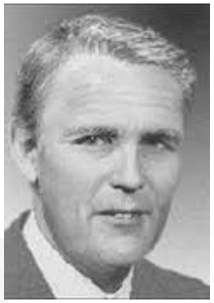
Kalman Filter Method

Kalman Filter method



- Updates the track parameters at each measurement point
- Takes into account evolution of state vector describing the track
  → multiple scattering
  → energy loss

What is a Kalman Filter



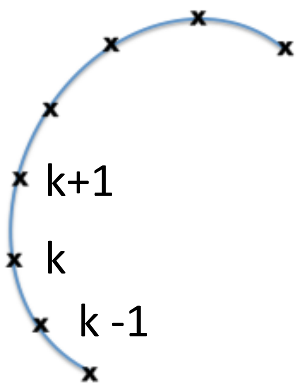
The Kalman filter, also known as linear quadratic estimation (wikipedia), is an algorithm that uses a series of measurements observed which contain noise (random variations i.e. MS) and other errors.

It produces estimates of unknown variables that tend to be more precise than those based on a single measurement alone and it produces a statistically optimal estimate of the underlying system state.

The filter is named for Rudolf E. Kálmán, the primary authors of the fitter.

Method to derive the best estimate of track parameters at a given point along the path by making use of all information collected at multiple measurement sites.

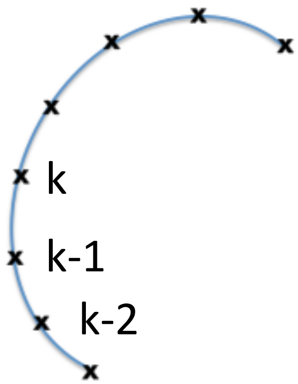
Components of Kalman Filter



1. A vector describing the particle at a particular location *(e.g k-1)* in the tracker (i.e. measurement site): $\mathbf{a}_{k-1}$
2. Equation of motion describing the evolution of state from one site to the next: $f(\mathbf{a}_{k-1}) = \mathbf{a}_k$
   – deterministic function ignores process noise ($\mathbf{w}_{k-1}$) with covariance matrix ($Q_{k-1} = \langle \mathbf{w}_{k-1} \mathbf{w}^{T}_{k-1} \rangle$), incorporated in state covariance matrix
3. A measurement vector: observables at a given site (e.g. midplane wire position)

   – assumes measurement error (ε_{k-1}) is unbiased; covariance matrix ($V_{k-1} = \langle \varepsilon_{k-1} \varepsilon^{T}_{k-1} \rangle$)
4. A function mapping the prediction at a given site onto the measurement: $h(\mathbf{a}_{k-1})$

How does it work?



- Goal to update the predicted state vector at site k: $\boldsymbol{a}_k^{k-1}$, using all measurements *up to site k* → $\boldsymbol{a}_k^*$
  - Let $(\chi^2)_{k-1}^{k-1}$ be the χ^2 up to site k-1
  - The χ^2 including the measurement and covariance matrix propagation to site k is: $(\chi^2)_k^k = (\chi^2)_{k-1}^{k-1} + \chi_+^2$
  - The optimal value of the state at site k is obtained by minimizing the χ^2 increment: $\frac{\partial \chi_+^2}{\partial \boldsymbol{a}_k^*} = 0$
  - where: $$\chi_+^2 = (\boldsymbol{a}_k^* - \boldsymbol{a}_k^{k-1})^T \left(\boldsymbol{C}_k^{k-1}\right)^{-1} (\boldsymbol{a}_k^* - \boldsymbol{a}_k^{k-1}) + (\boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^*))^T \boldsymbol{G} (\boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^*))$$

(G = inverse measurement error matrix)

  - yields ***filtered*** state:

Kalman Gain

$$\boldsymbol{a}_k = \boldsymbol{a}_k^{k-1} + \boxed{\left[\left(\boldsymbol{C}_k^{k-1}\right)^{-1} + \boldsymbol{H}_k^T \boldsymbol{G}_k \boldsymbol{H}_k\right]^{-1} \boldsymbol{H}_k^T \boldsymbol{G}_k} \left(\boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^{k-1})\right)$$

*Ref. Extended Kalman Filter Keisuke Fujii [The ACFA-Sim-J Group]

The Kalman Filter Equations

1) The Propagator

state propagator

$$\boldsymbol{a}_k^{k-1} = \boldsymbol{f}_{k-1}(\boldsymbol{a}_{k-1}^{k-1}) = \boldsymbol{f}_{k-1}(\boldsymbol{a}_{k-1}).$$

Predict what the state at site k is based on information up to site k-1

- A site is a location in the detector where a measurement is taken (e.g. plane)
- Energy loss correction in state propagation

propagator matrix

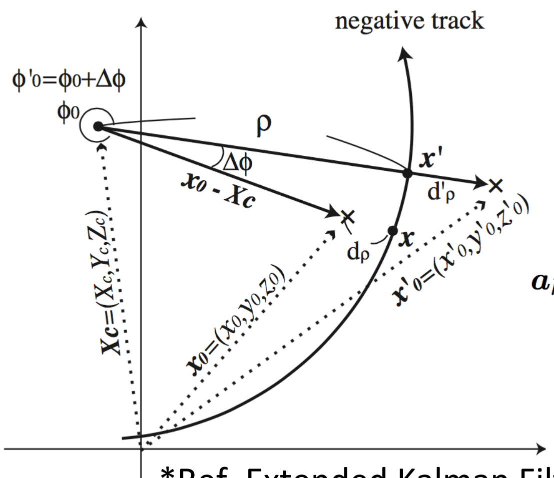
process noise (Multiple Scattering)

$$\boldsymbol{C}_k^{k-1} = \boldsymbol{F}_{k-1}\boldsymbol{C}_{k-1}\boldsymbol{F}_{k-1}^T + \boldsymbol{Q}_{k-1},$$

$$\boldsymbol{F}_{k-1} \equiv \left(\frac{\partial \boldsymbol{f}_{k-1}}{\partial \boldsymbol{a}_{k-1}}\right) = \begin{pmatrix} \frac{\partial d'_\rho}{\partial \boldsymbol{a}} \\ \frac{\partial \phi'_0}{\partial \boldsymbol{a}} \\ \frac{\partial \kappa'}{\partial \boldsymbol{a}} \\ \frac{\partial d'_z}{\partial \boldsymbol{a}} \\ \frac{\partial \tan\lambda'}{\partial \boldsymbol{a}} \end{pmatrix}$$

Propagate the Covariance matrix to site k is based on information up to site k-1

* e.g. for a helical track *(i.e track fitting in ~ constant B-field)*



$$\boldsymbol{a}' \equiv \boldsymbol{a}_k^{k-1} = (d'_\rho, \phi'_0, \kappa', d'_z, \tan\lambda')^T = \boldsymbol{f}_{k-1}(\boldsymbol{a}_{k-1})$$

$$\boldsymbol{a}_{k-1} \equiv \boldsymbol{a} = (d_\rho, \phi_0, \kappa, d_z, \tan\lambda)^T$$

$$\left\{\begin{array}{lcl} d'_\rho &=& (X_c - x'_0)\cos\phi'_0 + (Y_c - y'_0)\sin\phi'_0 - \frac{\alpha}{\kappa} \\ \phi'_0 &=& \left\{\begin{array}{ll} \tan^{-1}\left(\frac{Y_c - y'_0}{X_c - x'_0}\right) & (\kappa > 0) \\ \tan^{-1}\left(\frac{y'_0 - Y_c}{x'_0 - X_c}\right) & (\kappa < 0) \end{array}\right. \\ \kappa' &=& \kappa \\ d'_z &=& z_0 - z'_0 + d_z - \left(\frac{\alpha}{\kappa}\right)(\phi'_0 - \phi_0)\tan\lambda \\ \tan\lambda' &=& \tan\lambda, \end{array}\right.$$

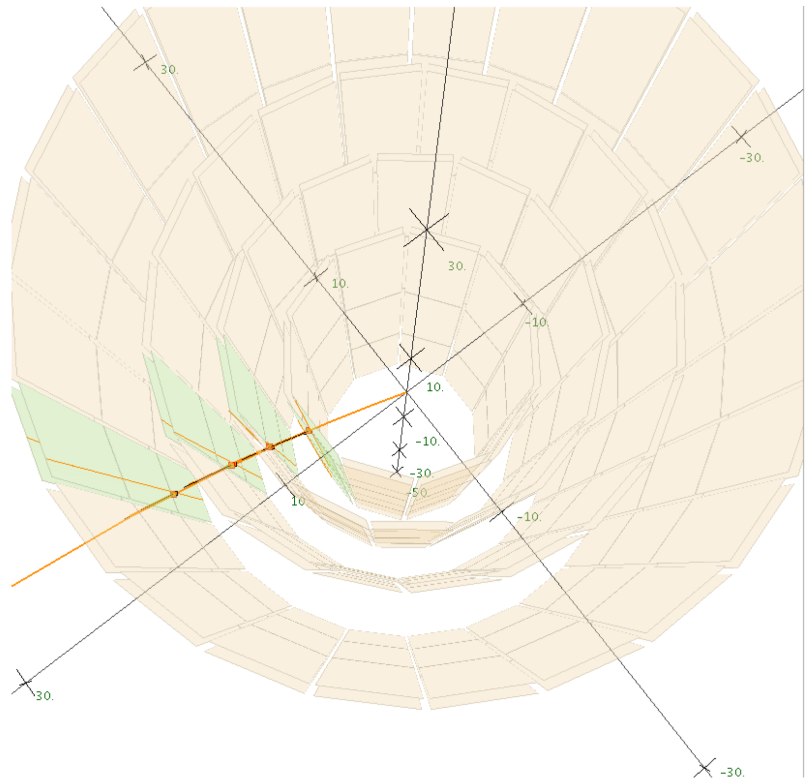
$$\left\{\begin{array}{lcl} X_c &\equiv& x_0 + \left(d\rho + \frac{\alpha}{\kappa}\right)\cos\phi_0 \\ Y_c &\equiv& y_0 + \left(d\rho + \frac{\alpha}{\kappa}\right)\sin\phi_0\,. \end{array}\right.$$

*Ref. Extended Kalman Filter Keisuke Fujii [The ACFA-Sim-J Group]

The Kalman Filter Equations
2) The Measurement

◆ Define the measurement a given site

- e.g. for a track in the BST
  - Cross coordinate (x,y,z) → up to 4 measurements
  - Strip number → up to 8 measurements



- **The greater the number of measurements the better the filter will work!**

The Kalman Filter Equations
3) The Projector

Predict the measurement at site k is based on the state vector at site k

1. Calculate the intersection of the track with the measurement surface (e.g. Si plane) at site k ($x_k(a)$)
2. Obtain the relation between the measurement and $x_k(a)$

state projector

$$h_k(a_k^{k-1}) = fcn\begin{pmatrix} x \\ y \\ z \end{pmatrix}_k$$

projector matrix

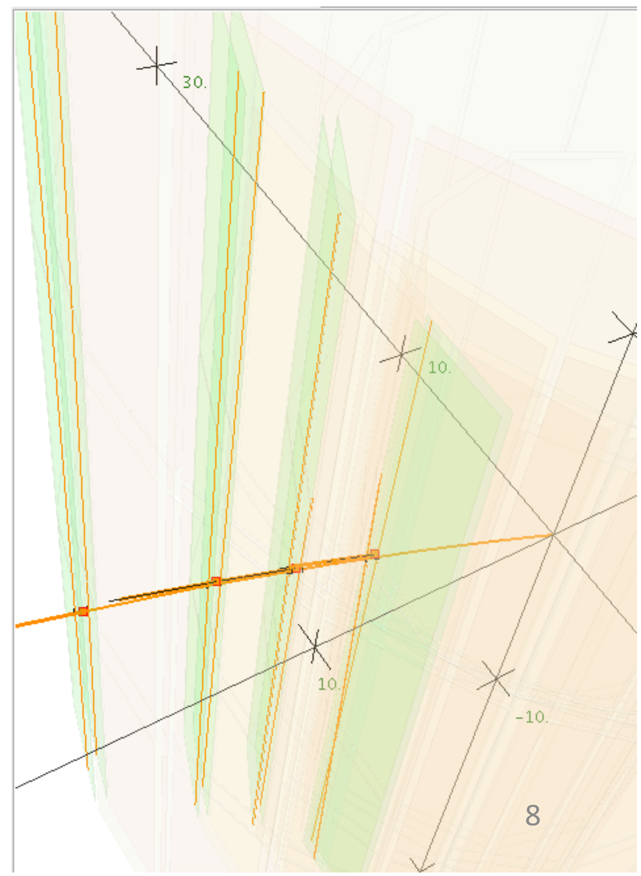
$$\boldsymbol{H}_k = \left(\frac{\partial \boldsymbol{h}_k}{\partial \boldsymbol{a}_k^{k-1}}\right)$$

Propagate the Measurement error matrix to site k

$$\boldsymbol{x}_k(\boldsymbol{a}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} = x_0 + d_\rho \cos\phi_0 + \frac{\alpha}{\kappa}(\cos\phi_0 - \cos(\phi_0+\phi)) \\ = y_0 + d_\rho \sin\phi_0 + \frac{\alpha}{\kappa}(\sin\phi_0 - \sin(\phi_0+\phi)) \\ = z_0 + d_z - \frac{\alpha}{\kappa}\tan\lambda\cdot\phi, \end{matrix}$$

(x_o, y_o, z_o) = reference point along the BST strip

$$\boldsymbol{h}_k(\boldsymbol{a}) = \boldsymbol{m}_k(\boldsymbol{x}_k(\boldsymbol{a}))$$ = strip index as a function of x,y,z



*Ref. Extended Kalman Filter Keisuke Fujii [The ACFA-Sim-J Group]

The Kalman Filter Equations

4) The Gain

Filtering of the state vector

$$\boldsymbol{a}_k = \boldsymbol{a}_k^{k-1} + \boldsymbol{K}_k \left(\boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^{k-1}) \right), \quad \boldsymbol{C}_k = \left[\left(\boldsymbol{C}_k^{k-1} \right)^{-1} + \boldsymbol{H}_k^T \boldsymbol{G}_k \boldsymbol{H}_k \right]^{-1}$$

Kalman ***Gain*** matrix

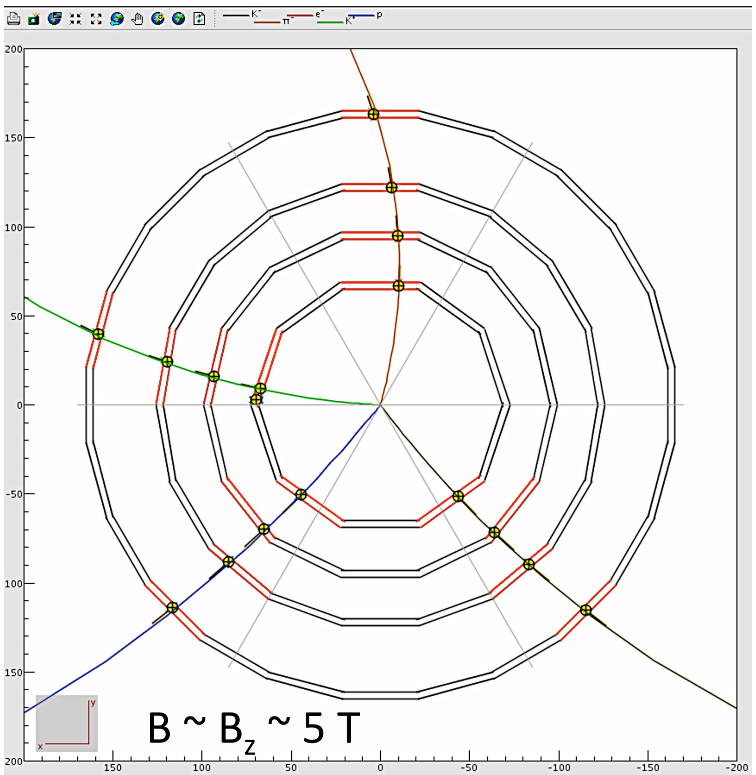
$$\boldsymbol{K}_k = \left[\left(\boldsymbol{C}_k^{k-1} \right)^{-1} + \boldsymbol{H}_k^T \boldsymbol{G}_k \boldsymbol{H}_k \right]^{-1} \boldsymbol{H}_k^T \boldsymbol{G}_k.$$

APPLICATIONS TO CLAS12 TRACKING

Tracking in CLAS12 SVT

1) The State Vector

Track parameters and Covariance matrix estimated from Global Fitting method prior to starting Kalman Filter

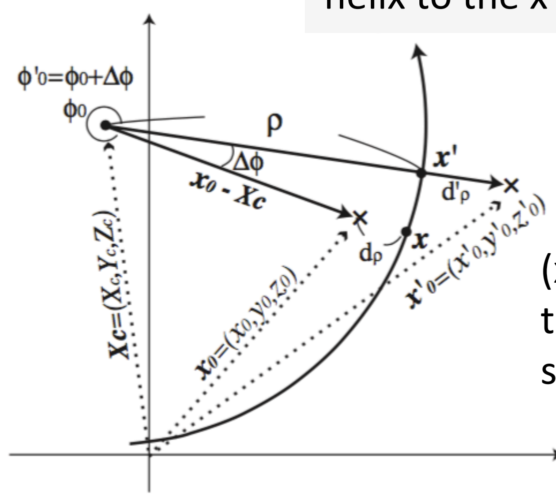


Event display of reconstructed tracks in CLAS12 SVT

- **site**: SVT module layer where a strip or cluster of strip fired (k = 1…8)
- **state**: 5-parameter helical track representation

$$a_k = \begin{pmatrix} d_\rho \\ \phi_0 \\ \kappa \\ d_z \\ \tan\lambda \end{pmatrix}_k$$

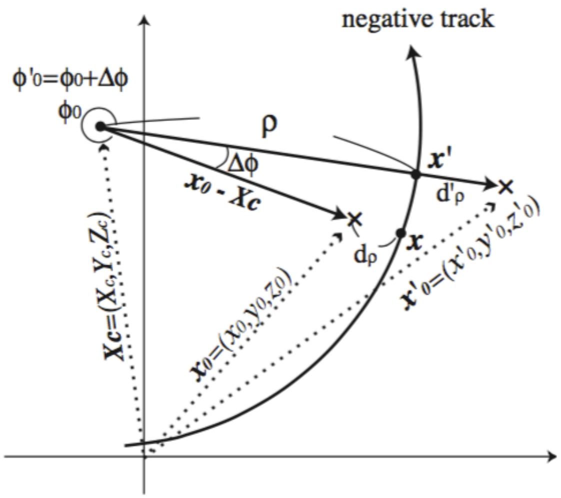
d_ρ: distance between the helix and the site ref. point (x_0,y_0,z_0) in x-y plane
ϕ_0: azimuthal angle of the ref. wrt helix center
$\kappa \equiv Q/Pt$
dz: distance between helix and reference point in the z direction
tan λ :dip angle, i.e., the angle of the helix to the x-y plane.



(x_0,y_0,z_0): intersection of the track with the signal strip or centroid

Tracking in CLAS12 SVT

2) The Propagator



(x_0,y_0,z_0): intersection of the track with the signal strip or centroid

- **Propagate from initial state estimated at DOCA to beam line to outermost SVT layer**
- **<u>state propagator</u>**: follows equations of motion for helical track $\boldsymbol{a}' \equiv \boldsymbol{a}_k^{k-1} = (d'_\rho, \phi'_0, \kappa', d'_z, \tan\lambda')^T = \boldsymbol{f}_{k-1}(\boldsymbol{a}_{k-1})$

$$\begin{cases} d'_\rho &= (X_c - x'_0)\cos\phi'_0 + (Y_c - y'_0)\sin\phi'_0 - \frac{\alpha}{\kappa} \\ \phi'_0 &= \begin{cases} \tan^{-1}\left(\frac{Y_c - y'_0}{X_c - x'_0}\right) & (\kappa > 0) \\ \tan^{-1}\left(\frac{y'_0 - Y_c}{x'_0 - X_c}\right) & (\kappa < 0) \end{cases} \\ \kappa' &= \kappa \\ d'_z &= z_0 - z'_0 + d_z - \left(\frac{\alpha}{\kappa}\right)(\phi'_0 - \phi_0)\tan\lambda \\ \tan\lambda' &= \tan\lambda, \end{cases}$$

$$\begin{cases} X_c &\equiv x_0 + \left(d\rho + \frac{\alpha}{\kappa}\right)\cos\phi_0 \\ Y_c &\equiv y_0 + \left(d\rho + \frac{\alpha}{\kappa}\right)\sin\phi_0 \,. \end{cases}$$

- **<u>state covariance matrix propagator</u>**: Jacobian of state projector

$$\boldsymbol{F}_{k-1} \equiv \left(\frac{\partial \boldsymbol{a}'}{\partial \boldsymbol{a}}\right) = \begin{pmatrix} \frac{\partial d'_\rho}{\partial \boldsymbol{a}} \\ \frac{\partial \phi'_0}{\partial \boldsymbol{a}} \\ \frac{\partial \kappa'}{\partial \boldsymbol{a}} \\ \frac{\partial d'_z}{\partial \boldsymbol{a}} \\ \frac{\partial \tan\lambda'}{\partial \boldsymbol{a}} \end{pmatrix}$$

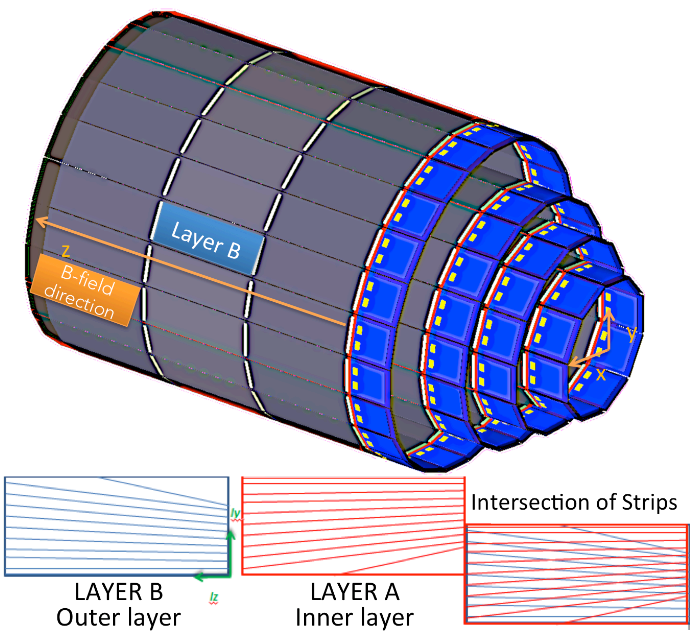
*Eqts. from paper
Extended Kalman Filter Keisuke Fujii
[The ACFA-Sim-J Group]

Tracking in CLAS12 SVT

3) The Measurement

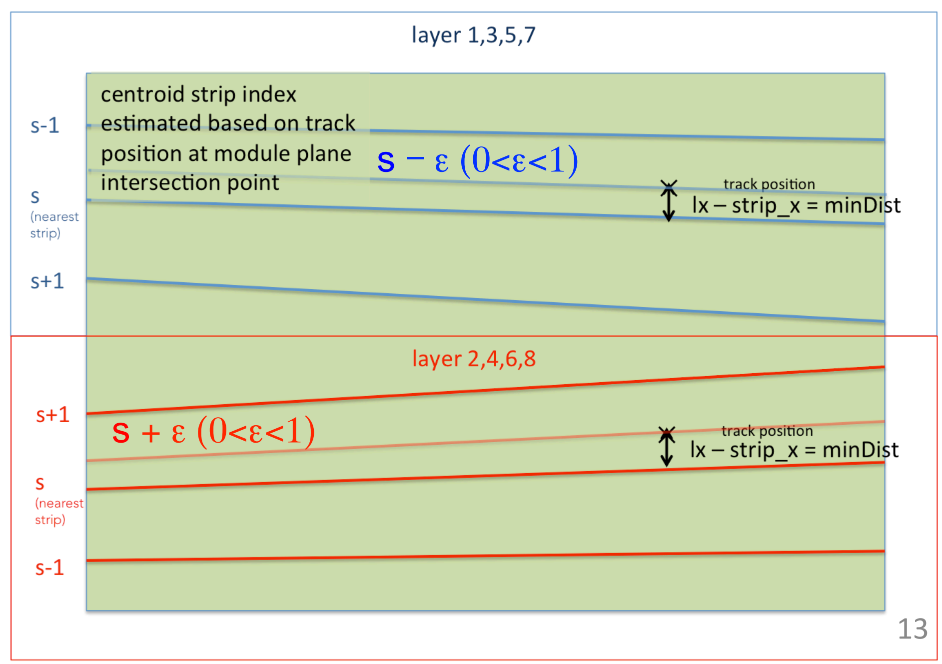
- Measurement: strip or cluster of strips centroid; error: strip/cluster resolution
- Projector $(\boldsymbol{h_k}(\boldsymbol{a}))$ = estimated centroid value based on track parameters at measurement site:

$$\begin{cases} x = x_0 + d_\rho \cos\phi_0 + \frac{\alpha}{\kappa}(\cos\phi_0 - \cos(\phi_0+\phi)) \\ y = y_0 + d_\rho \sin\phi_0 + \frac{\alpha}{\kappa}(\sin\phi_0 - \sin(\phi_0+\phi)) \\ z = z_0 + d_z - \frac{\alpha}{\kappa}\tan\lambda\cdot\phi, \end{cases}$$



SVT Geometry:

256 strips at 156 μm pitch oriented at graded angle from 0 to 3 deg.



Tracking in CLAS12 SVT

4) The Projector Matrix

Projector matrix calculation:

$$H = \frac{\partial c_k}{\partial X_k}\frac{\partial X_k}{\partial a_k}$$

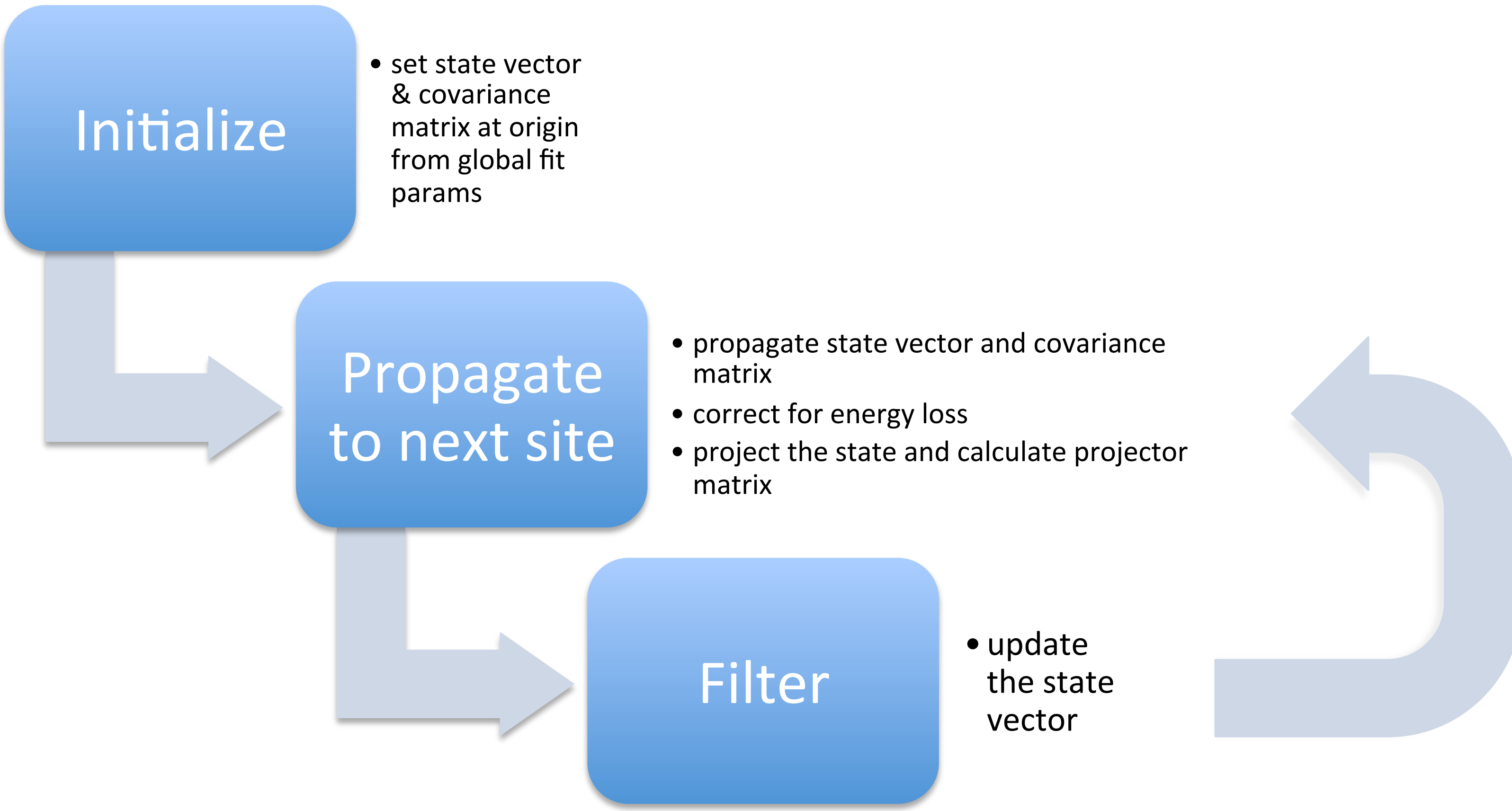
a_k : the state vector at site k

$X_k = (x, y, z)$: the projected track (i.e. a_k) intersection point with site k plane

$$\begin{cases} x = x_0 + d_\rho \cos\phi_0 + \frac{\alpha}{\kappa}(\cos\phi_0 - \cos(\phi_0 + \phi)) \\ y = y_0 + d_\rho \sin\phi_0 + \frac{\alpha}{\kappa}(\sin\phi_0 - \sin(\phi_0 + \phi)) \\ z = z_0 + d_z - \frac{\alpha}{\kappa}\tan\lambda \cdot \phi, \end{cases}$$

c_k : the centroid estimated from $X_k = (x, y, z)$

Sequence of Steps



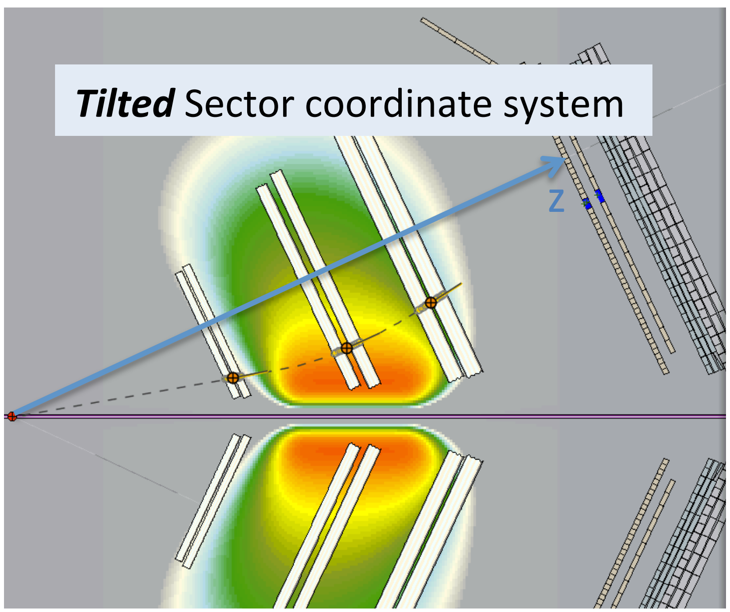
- ◆ Start process by propagating the state vector from innermost to outermost SVT module planes
- ◆ Once last layer reached, reverse direction and propagate to ref. point = (0,0,0)

❖ Code in validation stage

Tracking in CLAS12 DC

1) The State Vector

p and $\int B\,dl$ estimated from pattern recognition prior to fitting; global fitting method problematic due to inhomogeneity in the field.



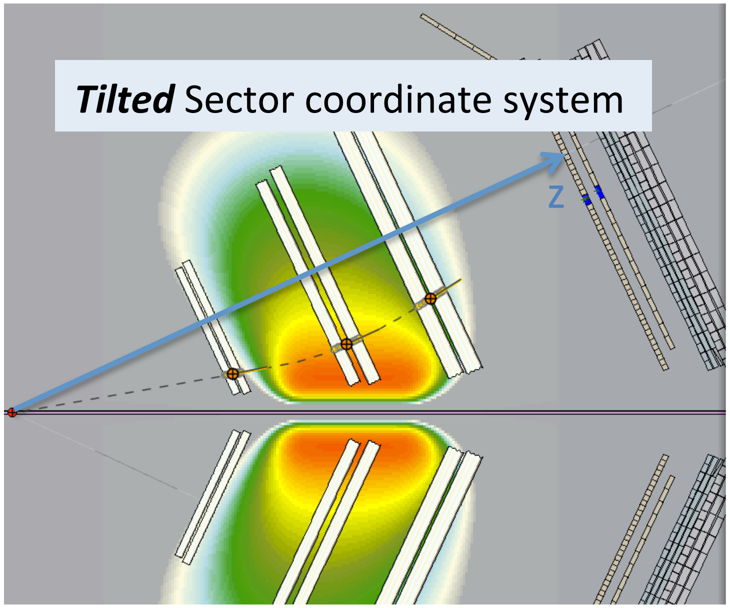
Event display of reconstructed tracks in CLAS12 DC

- **site**: DC layer plane where a fired (k = 1…36);
  - in tilted coordinate system, planes are perpendicular to z, so measurement sites are equidistant
- **state**: 5-parameter track representation

$$\tilde{x}(z) = \begin{pmatrix} x \\ y \\ t_x \\ t_y \\ q \end{pmatrix}, \quad \begin{aligned} t_x &= p_x/p_z \\ t_y &= p_y/p_z \\ q &= Q_e/\left|\vec{p}\right| \end{aligned}$$

Tracking in CLAS12 DC

2) The Propagator



- **Propagate from initial state estimated at layer 1 to outermost layer in DC sector**
- **<u>state propagator</u>**
  - Solve equation of motion directly assuming B is constant over a small enough step size δz

$$
\begin{aligned}
x(z) &= x_0 + t_{x0} \cdot s + \tfrac{1}{2} \cdot q_0 \cdot v \cdot A_x \cdot s^2 \quad , \\
y(z) &= y_0 + t_{y0} \cdot s + \tfrac{1}{2} \cdot q_0 \cdot v \cdot A_y \cdot s^2 \\
t_x(z) &= t_{x0} + q_0 \cdot v \cdot A_x \cdot s \quad , \\
t_y(z) &= t_{y0} + q_0 \cdot v \cdot A_y \cdot s \quad .
\end{aligned}
$$

$$
\begin{aligned}
A_x &= (1 + t_x^2 + t_y^2)^{\frac{1}{2}} \cdot [t_y \cdot (t_x B_x + B_z) - (1 + t_x^2) B_y] \quad , \\
A_y &= (1 + t_x^2 + t_y^2)^{\frac{1}{2}} \cdot \left[-t_x \cdot (t_y B_y + B_z) + (1 + t_y^2) B_x\right]
\end{aligned}
$$

- **<u>state covariance matrix propagator</u>**: Jacobian of state projector

$$
\boldsymbol{F}_{k-1} \equiv \left(\frac{\partial \boldsymbol{a}'}{\partial \boldsymbol{a}}\right) = \begin{pmatrix} \frac{\partial d'_\rho}{\partial \boldsymbol{a}} \\ \frac{\partial \phi'_0}{\partial \boldsymbol{a}} \\ \frac{\partial \kappa'}{\partial \boldsymbol{a}} \\ \frac{\partial d'_z}{\partial \boldsymbol{a}} \\ \frac{\partial \tan \lambda'}{\partial \boldsymbol{a}} \end{pmatrix}
$$

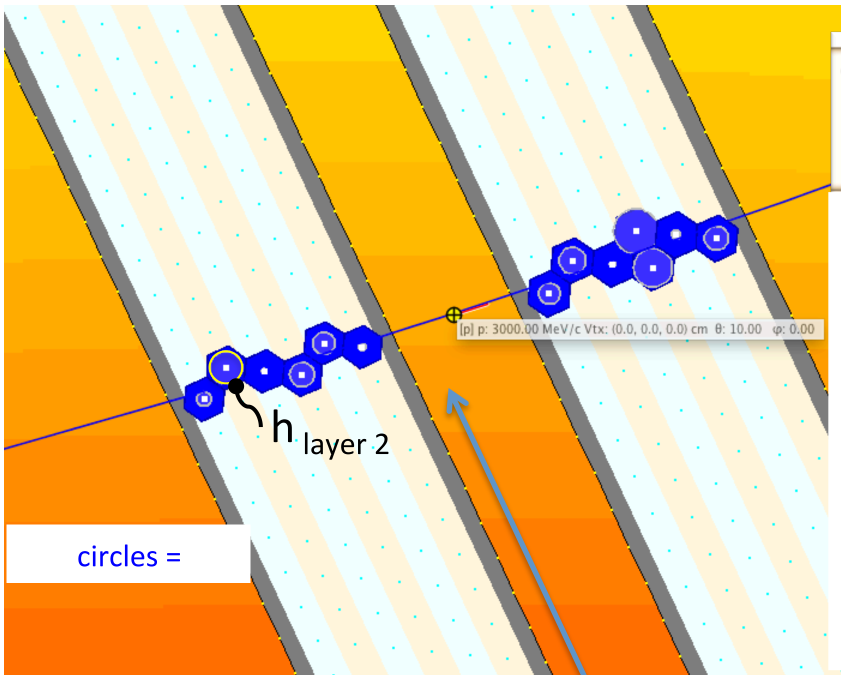
- <u>Ref</u>:

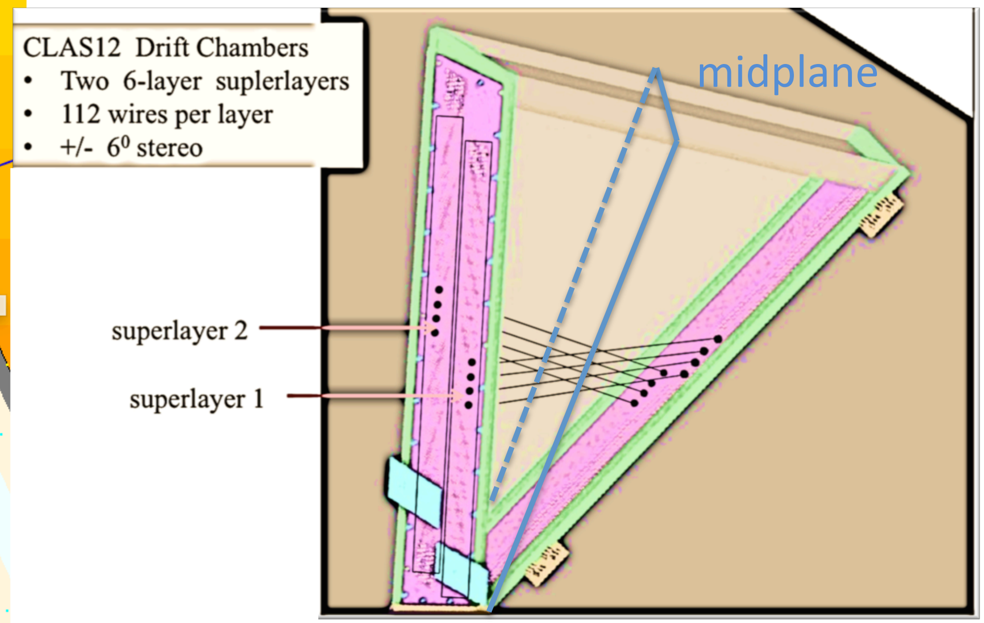
Optimized Integration of the Equations of Motion of a Particle in the HERA-B Magnet

Alexander Spiridonov
DESY Zeuthen / ITEP Moscow

Tracking in CLAS12 DC

3) The Measurement





Projection of state vector to the doca position @ midplane:

$$h_k(a_k^{k-1}) = (x)_k - \tan((sprlyr-1)\cdot stereoAngle)(y)_k$$

$sprlyr$ = superlayer, $stereoAngle = \pm 6$ degrees.

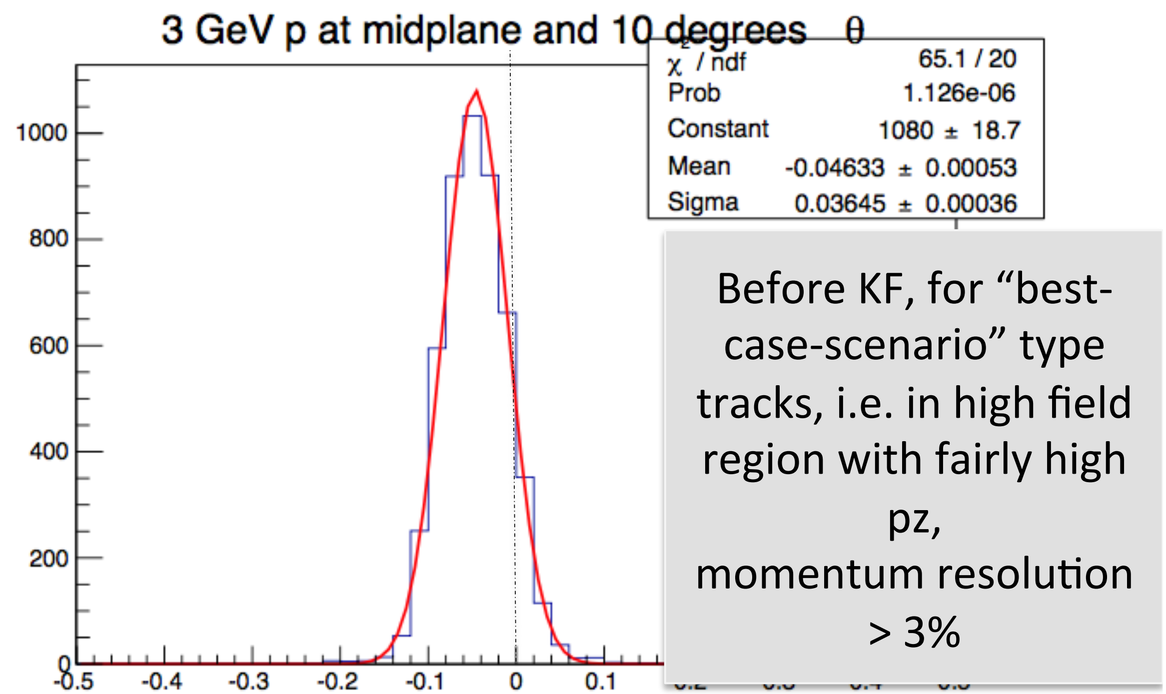
maps state onto an observable

Projector matrix:

$$\Rightarrow H = (1, -\tan((sprlyr-1)\cdot stereoAngle))$$

Does the Kalman Filter implemented in CLAS12 DC tracking work?

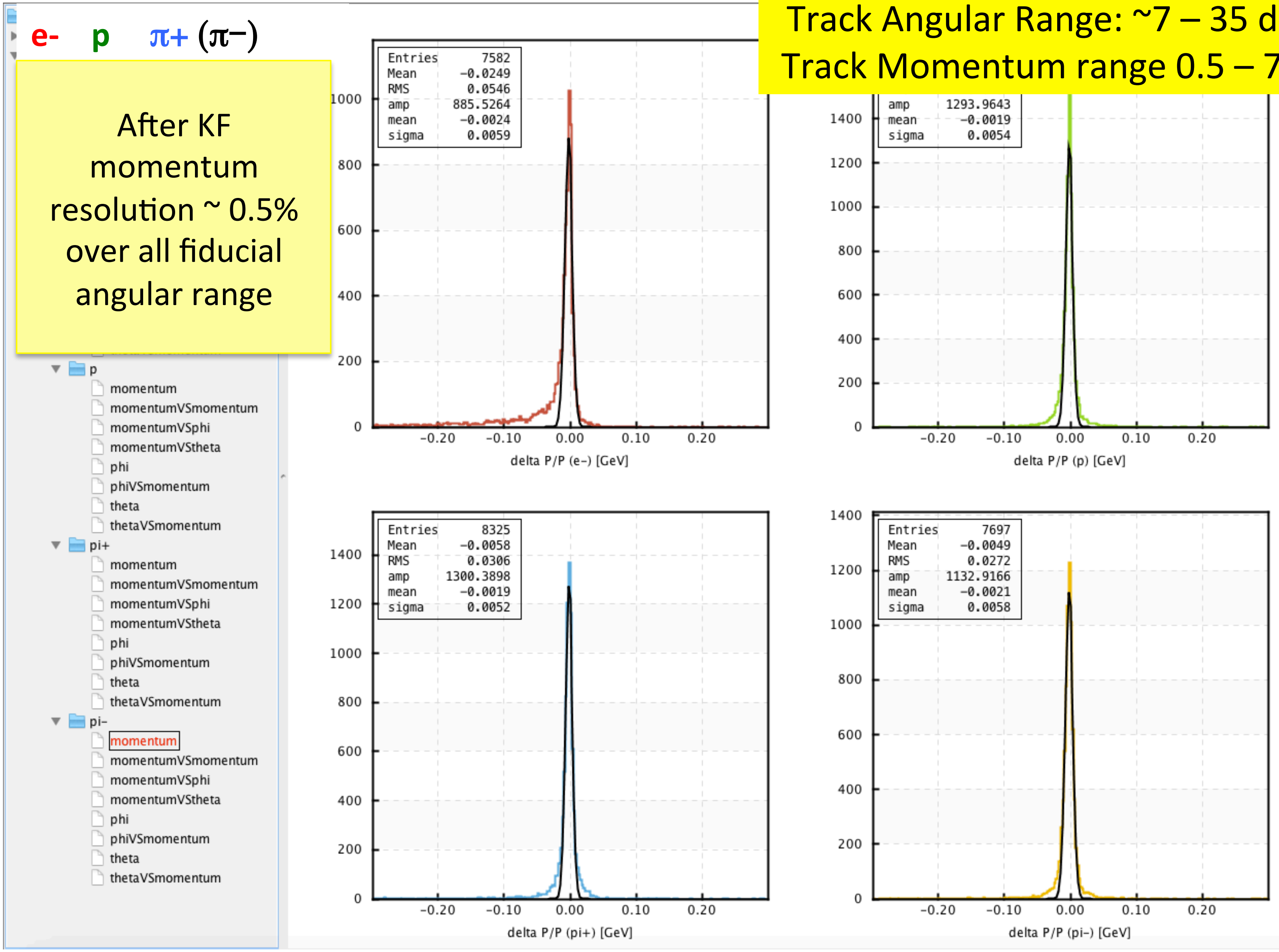
- Estimate track parameters in region 3
  - x
  - y
  - θ_X
  - θ_Y
  - q/p



- Inputs to Kalman Filter

DC Momentum Resolutions for 4-prong events

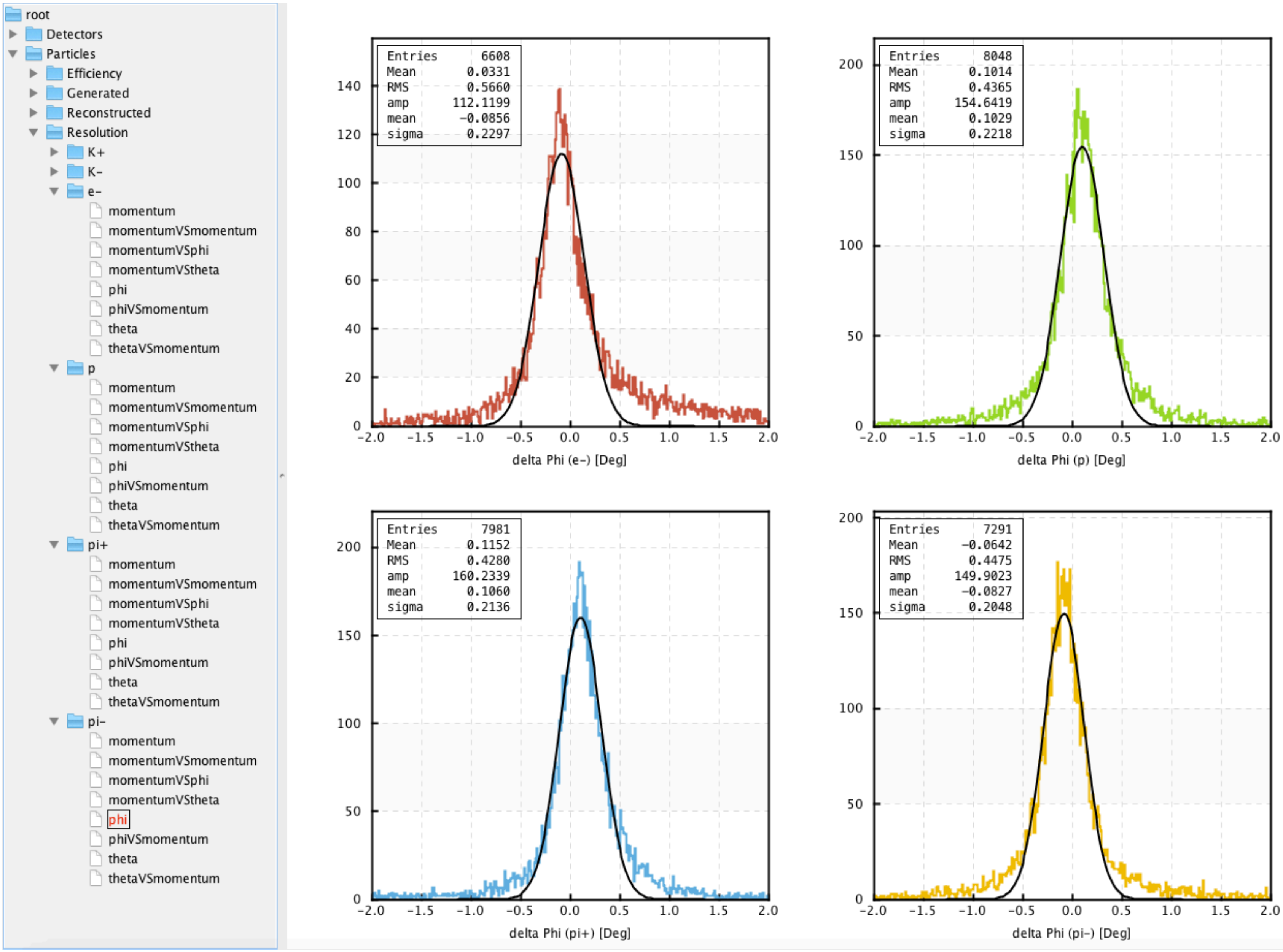
Track Angular Range: ~7 – 35 degrees
Track Momentum range 0.5 – 7 GeV/c



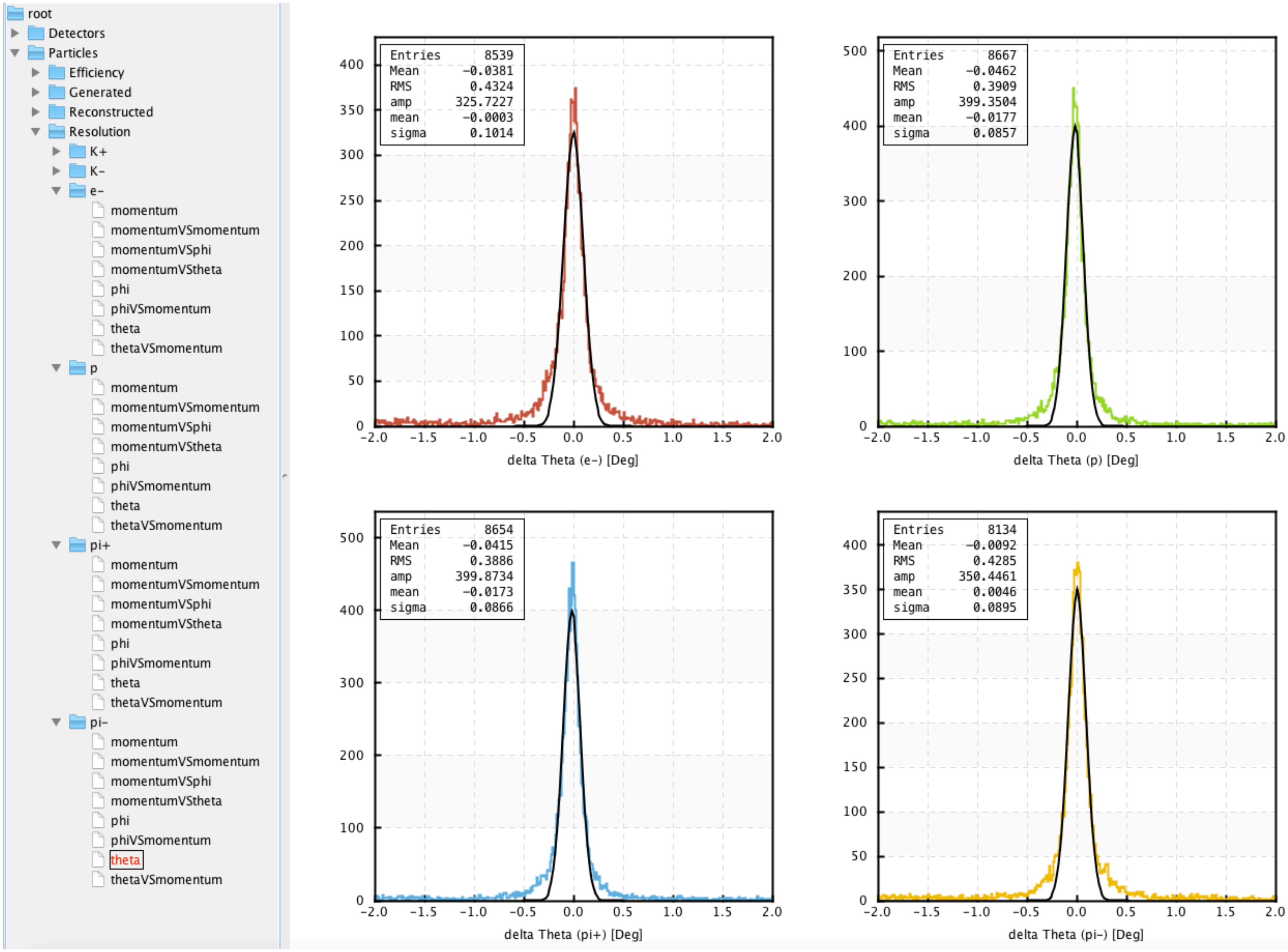
After KF momentum resolution ~ 0.5% over all fiducial angular range

DC Phi Resolutions

e- p π+ (π−)



DC Theta Resolutions e- p π+ (π−)



Concluding Remarks

I. The Kalman Filter is a standard track fitting method
II. A few important things to consider…
    I. choice of state vector & coordinate system
    II. choice of measurement vector
    III. A good estimate of the initial covariance matrix
III. Robustness of filter depends on number of measurements and initialization of parameters