PHYS 722/822 Lecture notes

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Free neutron decay

The free neutron decay is a reaction that produces a proton and leptons through the weak interaction:

$$n^0 \rightarrow p^+ + \beta^- + \bar{\nu}_e$$

To understand this weak decay, first we need to recall Fermi's Golden Rule, which is the connection between reaction rate, transition matrix, and the density of final states:

$$W = \frac{2\pi}{\hbar}\left|M_{fi}\right|^2 * \frac{dn}{dE'}$$

where W is the reaction rate, M_{fi} is the transition matrix, and $\frac{dn}{dE'}$ is the final density of states in the energy interval dE'. Modifying this rule for weak decay, the equation becomes:

$$dW = \frac{2\pi}{\hbar}\left|M_{fi}\right|^2 * \Delta\phi(E_e)$$

where $\Delta\phi(E_e)$ is the phase space. The phase space is related to the final density of states, with E_0 as the total energy available to decay, E_e as the electron energy, and E_ν as the neutrino energy:

$$\Delta\phi(E_e) = dn(p_e)dn(p_\nu)\delta(E_0 - E_e - E_\nu) = \left(\frac{V4\pi p_e^2 dp_e}{(2\pi\hbar)^3}\right)\left(\frac{V4\pi p_\nu^2 dp_\nu}{(2\pi\hbar)^3}\right)\delta(E_0 - E_e - E_\nu)$$
$$= \frac{(4\pi V)^2}{(2\pi\hbar)^6} p_e^2 dp_e p_\nu^2 dp_\nu \delta(E_0 - E_e - E_\nu)$$

The volume is in 6 dimensional phase space, with the total decay energy, E_0=(M_A - $M_{A'}$), fully relativistic (M_A and $M_{A'}$ are the initial and final nuclei). E_e and E_ν add up to the mass difference in the system, so the 4-momentums are conserved in the decay.

Next we use $pdp = EdE$ for relativistic E^2=p^2 +m^2 (this relation is also true for the nonrelativistic case) to express $\Delta\phi(E_e)$ in dE. Then we can do the integral over dE_ν because we want the answer in terms of E_0 and E_e,, which are the components that will be measured. The delta function tells us that E_ν=E_0-E_e.

$$\Delta\phi(E_e) = \frac{(4\pi V)^2}{(2\pi\hbar)^6} p_e E_e dE_e p_\nu E_\nu dE_\nu \delta(E_0 - E_e - E_\nu) \; \rightarrow \frac{(4\pi V)^2}{(2\pi\hbar)^6} p_e E_e dE_e p_\nu E_\nu$$

From here, we could replace p_ν with E_ν since the mass of the neutrino is negligible, but written out with the mass:

$$\frac{(4\pi V)^2}{(2\pi\hbar)^6} p_e E_e dE_e p_\nu E_\nu = \frac{(4\pi V)^2}{(2\pi\hbar)^6} p_e E_e dE_e \sqrt{E_\nu^2 - m_\nu^2} E_\nu \approx \frac{(4\pi V)^2}{(2\pi\hbar)^6} p_e E_e dE_e \left(E_\nu^2 - \frac{m_\nu^2}{2}\right)$$

where the last term was obtained with a Taylor expansion of the square root. This expansion is valid if we measure the spectrum of the emitted electron energy E_e and find that to first order M_{fi} is constant with respect to E_e. Replacing p_e and E_ν gives

$$\Delta\phi(E_e) \approx \frac{(4\pi V)^2}{(2\pi\hbar)^6} \sqrt{E_e^2 - m_e^2} E_e dE_e \left((E_0 - E_e)^2 - \frac{m_\nu^2}{2}\right)$$

Detecting the mass of the neutrino

It is actually quite difficult to verify the mass of the neutrino, although it is possible from the kinematics of β-decay. From the probability of measuring an electron in the interval E_e to $E_e + dE_e$, we can graph a Kurie Spectrum from the expression:

$$\sqrt{\frac{Prob(E_e \dots E_e + dE_e)}{p_e E_e}} \sim \sqrt{\left(E_\nu^2 - \frac{m_\nu^2}{2}\right) dE_e}$$

If m_ν=0, then the function will behave as E_0 - E_e, decreasing in a straight line as E_e increases, until it hits the E_e line at E_0. This means all the decay energy can be delivered to the electron. If the neutrino is not massless, then at high E_e, the line will bend until it crosses the E_e line vertically at $E_0 - m_\nu c^2$, exhibiting quadratic behavior that is only visible at a low count rate. This difference is caused by some of the total decay energy being given to the mass fluctuations of the neutrino.

However, since $E_e \sim MeV$ and $E_\nu \sim eV$, there is a large energy discrepancy. This makes it very hard to measure. It is also hard to measure because the graph's change in behavior only occurs when there is a low count rate. Even so, experiments like KATRIN try to measure the neutrino mass. They built a spectrometer to measure 18.6 keV electrons from the beta-decay of tritium. Even at this relatively low E_e, the sensitivity needed to measure m_ν would be very high.

Beta Decay of the Nucleus

When the nucleon is inside the nucleus, the expression for weak decay needs to be modified. The phase space $\Delta\phi(E_e)$ needs to be multiplied by the Fermi function $F(Z', E_e)$. This Fermi function helps explain how the emitted electron or positron will interact with the rest of the nucleus. When the nucleon decays inside the nucleus, the emitted e^- or e^+ does not appear far away from the nucleus. Since the Weak Interact is a zero length interaction, the emitted β particle will appear *inside* the nucleus. While this β particle will ignore the Strong Interaction because it is a lepton, it will still interact with the Coulomb potential created by the surrounding protons.

Recall that the Coulomb potential is a well that gets shallower as the distance from the nucleus increases by 1/r. This well will create a shift in the phase space depending on the charge of the lepton. Electrons will use the energy from where it was produced to give it a boost while it is inside, but the energy will fall back after it leaves the nucleus. A positron will have a lower energy inside and gain energy as it comes out of the nucleus.

Then the total decay rate will be:

$$W = \frac{2\pi}{\hbar}\left|M_{fi}\right|^2 \int_{E_e=0}^{E_0} \Delta\phi(E_e)F(Z',E_e)$$

The integral really only contains "accounting" information; all the real physics is stored in the transition matrix M_{fi}. Next, we can normalize every p and E to the electron mass m_e so that $\varepsilon = \frac{E}{m_e c^2}$, which will make everything dimensionless. This will lead to a factor of $(m_e c^2)^5$. Recalling that the decay rate is equal to the inverse of the lifetime of the particle, $\tau = \tau_{1/2}/\ln 2$:

$$W = \frac{1}{\tau} = \frac{2\pi}{\hbar}\left|M_{fi}\right|^2 (m_e c^2)^5 \int_1^{\varepsilon_0} \Delta\phi(\varepsilon_e)F(Z',\varepsilon_e)$$

We can then replace the integral with an f-factor:

$$f(Z',E_0) = \int_1^{\varepsilon_0} \sqrt{\varepsilon_e^2 - 1}\,\varepsilon_e d\varepsilon_e \left((\varepsilon_0 - \varepsilon_e)^2 - \frac{m_\nu^2}{2}\right)F(Z',\varepsilon_e)$$

Substituting this into the previous equation and multiplying both sides by $\tau_{1/2}/\ln 2$ gives

$$1 = \frac{m_e^5 c^4}{2\pi^3 \hbar^7 \ln 2}\left|M_{fi}\right|^2 f * \tau_{1/2}$$

where $f * \tau_{1/2}$ is called the ft value. The ft value conveys more distinctive information than the half-life because it also tells how the matrix elements behave. It doesn't explain them, but organizes the half-life. Sometimes it is recorded as $\log_{10}(ft)$ which ranges from approximately 3 to 22.

How to calculate M_{fi}?

The transition matrix elements can be calculated in two parts: vector and axial. Transitions that happen from the vector part are called Fermi decays. Transitions from the axial part are called Gamow-Teller decays.

The transition matrix is:

$$|M_{fi}| = g_v < f|\tau^{\pm}|i > \; + g_A < f|\tau^{\pm}\vec{\sigma}|i >$$

Where g_v and g_A are the vector and axial coupling constants, $\tau^{\pm}$ is the isospin ladder operator, and <f| and |i> are expressed in terms of the final and initial nucleons. If we simply approximate the nucleus as having only proton and neutron quark arrangements, then a β^- decay would look like:

$$|M_{fi}| = < \sum_{p}^{A\prime} \psi_p \,|g_v^N \tau^+| \sum_{n}^{A} \psi_n > \; + < \sum_{p}^{A\prime} \psi_p \,|g_A^N \tau^+ \vec{\sigma}| \sum_{n}^{A} \psi_n >$$

Where N in $g_{v/A}^N$ is for neutron, n and p sum over the nucleons in the initial and final nucleus with nucleon counts A and A', respectively. The isospin ladder operator turns the neutrons into protons for the β^- decay. This contains the wave function of the nucleons inside the nucleus, which tells us about the nuclear structure and the Weak Force. Since there is no angular momentum operator, the original neutron will convert to a proton that can be in the same or in a different orbit.

Decay selection rules

Below are the selection rules for the overall nuclear spin, J, and parity, π, that tell which decays are allowed.

Fermi transition (first term):

ΔS=0 because the operator has no angular momentum.

Δπ=0 because the operator has no parity.

ΔJ=0 for nuclear spin J.

Gamow-Teller transition (second term):

ΔS=0 or ΔS±1

ΔP=0

ΔJ=0, ±1

Can transition from 1 → 1, but not from 0 →0

Example: free neutron decay starts with neutron spin = 1/2 and ends with proton spin = 1/2. Since there is no spin change, both Fermi and Gamow-Teller can apply to this.

But it is also possible to violate these transition rules. These are called "forbidden" transitions. This is done by transferring orbital angular momentum L. You try to find the lowest level of orbital transfer needed to permit the transition.

For Fermi transition: ΔJ= ΔL. (ΔL will tell you how many times it is forbidden)

For Gamow-Teller transition: ΔJ= ΔL +1 (in the usual sense of angular momentum addition).

Example: Potassium-40 contributes to some of the radiation in the human body. The Potassium has $J^\pi = 4^-$. Then ^{40}K decays to calcium-40, which is a doubly magic nucleus. Calcium has $J^\pi = 0^+$, which is true for all doubly magic nuclei. To transfer from ^{40}K to ^{40}Ca, you need to change J,L four times! This is not possible as a Fermi transition because the parity would still be negative, since

$$\frac{p_f}{p_c} = (-1)^{\Delta L}$$

Hence, the only possibility is a 3-times forbidden Gamow-Teller transition.

Sometimes you can have transitions that are only Fermi or only Gamow-Teller, but if the selection rules for both are satisfied then both transitions will happen.

You can also have super allowed decays. This happens when the initial and final wave functions (including spin) have a perfect overlap. This means that after a nucleon decays, the transitioned nucleon will have the same quantum numbers that it had before. This means the corresponding nuclear states between which the transition occurs are in the same isospin multiplet. The ft values of these transitions are roughly the same as the ft value of a free neutron. Most of these transitions are β^+ decays.

Example: Mirror Nuclei

Consider the β^- decay of tritium into helium-3:

$$^3_1H \rightarrow {}^3_2He + \beta^- + \bar{\nu}_e$$

Tritium and helium-3 have basically the same wave equation; the only difference is an exchange of nucleons. Tritium has 2 neutrons and one proton, so to transfer to helium-3, the isospin of one of the neutrons is increased, releasing an electron and an antielectron neutrino. The new proton will have the same quantum numbers that it had as a neutron, so this will be a super allowed transition. Besides the free neutron, this is the only other exception to the rule that protons decay to neutrons for super allowed transitions.

Example: Isospin Triplet

Listed in order of energy, we have N^{14} ground state with $J^\pi = 1^+$, C^{14} ground state with 0^+, N^{14} excited state with 0^+, and O^{14} with 0^+. Since O^{14} has higher energy than the N^{14} excited state, this transition can occur. It is a $0^+ \rightarrow 0^+$ transition, so it must be a pure Fermi decay. Then it can decay to C^{14} which is also a Fermi decay. These two transitions are super allowed because protons in the nucleus β^+ decay into neutrons of the same shell level ($1p_{1/2}$).

Because the ground state of C^{14} is only above the ground state of N^{14}, that is the only place it could transition to in this triplet. However, this would not be super allowed because of the spin change. This is an allowed Gamow-Teller decay because ΔL=1 (the nucleon flips its spin). Thankfully, this transition takes a very long time ($\tau_{1/2}$ = 5730 years). It allows for carbon dating of objects older than 6000 years, which permits us to place objects into Earth's long geological history.

Why do forbidden transitions take so long?

For the forbidden transition to occur, something must take away the change in the total angular momentum ΔJ. This cannot occur for Fermi decay (ΔJ=0) or Gamow-Teller decay (ΔJ=0,±1). You need a $\beta^\pm$ and ν_e to take more than 1 angular momentum unit to have a forbidden transition.

To see how orbital angular momentum could do this, let's approximate the wave of the electron as a plane wave.

$$\psi(x) \sim e^{ikr}$$

This can be Taylor expanded to

$$\psi(x) \sim 1 + ikr - \frac{kr^2}{2} + \cdots$$

Or decomposed as

$$\psi(x) \cong \sum c_e\, Y_{l0}(\theta) j_l(kr) \approx \sum c_e\, Y_{l0}(\theta)(kr)^l \quad for\ kr \ll 1$$

Here, r is the size of a nucleon, or a nucleus at most (so ~fm) and k is on the order of MeV's. This means $kr \sim 10^{-2}$. So the higher l is, the more suppressed the decay is. The ft value is related to the inverse square of the transition matrix, which squares the suppression of l. Since the ft value is directly related to the half-life, we see that this will increase the time it takes for the transition to happen, making it live longer.

Further reading in Povh 7th edition:

Chapter 4.3: Fermi's "Golden Rule"

Chapter 16.6: Beta decay of the Neutron, transition matrix elements, neutron lifetime

Chapter 18.6: Beta Decay of the Nucleus, decay selection rules, neutrino mass