**Nuclear Physics - Problem Set 8 – Solution**

**Problem 1)**

a) The first part of the given wave function of course fulfills the equation $H_0 e^{i\vec{k}\cdot\vec{r}} = -\hbar^2 \frac{\vec{\nabla}^2}{2m} e^{i\vec{k}\cdot\vec{r}} = \frac{\vec{p}^2}{2m} e^{i\vec{k}\cdot\vec{r}} = E e^{i\vec{k}\cdot\vec{r}}$ , as is well known from elementary Quantum Mechanics. For the second part, we use the expression for the Laplacian in spherical coordinates as given in the problem:

$$H_0\, f(\theta,\varphi)\frac{e^{ikr}}{r} = -\hbar^2 \frac{\vec{\nabla}^2}{2m} f(\theta,\varphi)\frac{e^{ikr}}{r} =$$

$$= -\frac{\hbar^2}{2m}\left[ f(\theta,\varphi)\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\frac{e^{ikr}}{r} + \left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f(\theta,\varphi)}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f(\theta,\varphi)}{\partial\varphi^2}\right)\frac{e^{ikr}}{r}\right]$$

$$= -\frac{\hbar^2}{2m}\left[ f(\theta,\varphi)\frac{1}{r^2}\frac{\partial}{\partial r}\left((ikr-1)e^{ikr}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f(\theta,\varphi)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f(\theta,\varphi)}{\partial\varphi^2}\right)\frac{e^{ikr}}{r}\right]$$

$$= -\frac{\hbar^2}{2m}\left[ f(\theta,\varphi)\frac{1}{r^2}\left(ik + (ikr-1)ik\right)e^{ikr} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f(\theta,\varphi)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f(\theta,\varphi)}{\partial\varphi^2}\right)\frac{e^{ikr}}{r}\right]$$

$$= \frac{1}{2m}p^2 f(\theta,\varphi)\frac{1}{r}e^{ikr} - \frac{\hbar^2}{2mr^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f(\theta,\varphi)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f(\theta,\varphi)}{\partial\varphi^2}\right)\frac{e^{ikr}}{r}$$

The first part of the last expression is indeed of the form $H\psi=E\psi$ with the same eigenvalue E, while the second part falls off with an additional factor $1/r^2$ and therefore becomes negligible in the limit $r \to \infty$, as long as the angular derivatives inside the parentheses are well-behaved (finite).

b) This is a simple plug-in problem:

$$j_r = \frac{h}{2mi}\left[\psi^*\left(\frac{\partial}{\partial r}\psi\right) - \left(\frac{\partial}{\partial r}\psi^*\right)\psi\right] =$$

$$= \frac{h}{2mi}\left[f^*(\theta,\varphi)\frac{e^{-ikr}}{r}\left(\frac{\partial}{\partial r}f(\theta,\varphi)\frac{e^{ikr}}{r}\right) - \left(\frac{\partial}{\partial r}f^*(\theta,\varphi)\frac{e^{-ikr}}{r}\right)f(\theta,\varphi)\frac{e^{ikr}}{r}\right]$$

$$= \frac{h}{2mi}|f(\theta,\varphi)|^2\left[\frac{e^{-ikr}}{r}\left(\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right) - \left(\frac{\partial}{\partial r}\frac{e^{-ikr}}{r}\right)\frac{e^{ikr}}{r}\right]$$

$$= \frac{h}{2mi}|f(\theta,\varphi)|^2\left[\frac{e^{-ikr}}{r}\left((ikr-1)\frac{e^{ikr}}{r^2}\right) - \left((-ikr-1)\frac{e^{-ikr}}{r^2}\right)\frac{e^{ikr}}{r}\right]$$

$$= \frac{2ipr}{2mi}\frac{1}{r^3}|f(\theta,\varphi)|^2 = \frac{p}{m}\frac{|f(\theta,\varphi)|^2}{r^2} = v|\psi|^2$$

**Problem 2)**

a) Since this is a non-relativistic situation, I find (with E = 2.5 MeV) $k_{out} = (2\mu c^2\, E)^{1/2}\, /\hbar c = (m_{proton}c^2\, 2.5\text{ MeV})^{1/2}\, /\hbar c = 48.4\text{ MeV}/\,\hbar c =$ $= 0.245\text{ fm}^{-1}$ (remember, $\mu$ is $m_{proton}/2$ here). The cross section is

$$\frac{d\sigma}{d\Omega} = |f_0(\theta)|^2 = \frac{1}{k^2}\sin^2(\delta_o) = 9.78\,fm^2 = 97.8\,mb$$

b) I use equation 17.7 on page 290 of Povh *et al*. with $k_{out} = (2\mu c^2\, E)^{1/2}\, /\hbar c =$ $0.245\text{ fm}^{-1}$ and a = 1.6 fm. Subtracting the last term on the rhs from both sides and taking the tan on both sides, I get a transcendental equation that I can solve for |V|. The numerical solution I find is
$|V_o| = 33.355$ MeV. (Just plug it in the equation to convince yourself that this is correct.)
This result is a very rough approximation, although it is at least the right order of magnitude (compare to the plot on page 291 - you'd have to average out the exponential, negative potential at r>0.8 fm and the repulsive core at smaller r).