Nuclear Physics - Problem Set 9 – Solution

Problem 1)

Following the hint, I first calculate the gradient of ψ:
$\vec{\nabla}\frac{e^{-mr}}{r} = \frac{\partial}{\partial r}\frac{e^{-mr}}{r}\hat{r} = \left(-\frac{1}{r^2} - \frac{m}{r}\right)e^{-mr}\hat{r}$. Using Gauß' theorem, I can integrate the negative divergence of this result (= the Laplacian of ψ) over a sphere of radius R by integrating the expression itself over the surface of the sphere:

$$-\iiint_{Volume} div\vec{\nabla}\frac{e^{-mr}}{r}d^3\vec{r} = -\iint_{Surface}\left(-\frac{1}{r^2} - \frac{m}{r}\right)e^{-mr}\hat{r}\cdot d\vec{a} = 4\pi(1+mR)e^{-mR}.$$

The first term of the r.h.s. (the delta-function) of course simply yields 4π, and the remaining term yields

$$-\iiint_{Volume} m^2\frac{e^{-mr}}{r}d^3\vec{r} = -4\pi m^2\int_0^R re^{-mr}dr = -4\pi m^2\left[r\frac{e^{-mr}}{-m}\right]_0^R + 4\pi m^2\int_0^R\frac{e^{-mr}}{-m}dr$$

$$= 4\pi m R e^{-mR} + 4\pi m^2\left[\frac{e^{-mr}}{m^2}\right]_0^R = 4\pi m R e^{-mR} + 4\pi e^{-mR} - 4\pi = 4\pi(1+mR)e^{-mR} - 4\pi$$

The last term cancels with the 4π from the delta-function, and both sides indeed turn out equal. Since ψ and therefore its derivatives are spherically symmetric, it follows that the l.h.s. and r.h.s. must be equal if their integrals over arbitrary spheres are equal.

Problem 2)

First, I rewrite the Tensor operator using all the hints given:

$$S_{12}=3\left(\sigma_1^+\left(\frac{x-iy}{r}\right)+\sigma_1^-\left(\frac{x+iy}{r}\right)+\sigma_1^z\frac{z}{r}\right)\left(\sigma_2^+\left(\frac{x-iy}{r}\right)+\sigma_2^-\left(\frac{x+iy}{r}\right)+\sigma_2^z\frac{z}{r}\right)-$$

$$-\left(2\sigma_1^+\sigma_2^-+2\sigma_1^-\sigma_2^++\sigma_1^z\sigma_2^z\right)$$

$$=3\left(\sigma_1^+\sin\theta\, e^{-i\varphi}+\sigma_1^-\sin\theta\, e^{+i\varphi}+\sigma_1^z\cos\theta\right)\left(\sigma_2^+\sin\theta\, e^{-i\varphi}+\sigma_2^-\sin\theta\, e^{+i\varphi}+\sigma_2^z\cos\theta\right)-$$

$$-\left(2\sigma_1^+\sigma_2^-+2\sigma_1^-\sigma_2^++\sigma_1^z\sigma_2^z\right)$$

$$=3\left[\sin^2\theta\left(\sigma_1^+\sigma_2^+\,e^{-2i\varphi}+\sigma_1^-\sigma_2^-\,e^{2i\varphi}+\sigma_1^+\sigma_2^-+\sigma_1^-\sigma_2^+\right)+\cos^2\theta\,\sigma_1^z\sigma_2^z\right]+$$

$$+3\sin\theta\cos\theta\left[\left(\sigma_1^+\sigma_2^z+\sigma_1^z\sigma_2^+\right)e^{-i\varphi}+\left(\sigma_1^-\sigma_2^z+\sigma_1^z\sigma_2^-\right)e^{+i\varphi}\right]-2\sigma_1^+\sigma_2^--2\sigma_1^-\sigma_2^+-\sigma_1^z\sigma_2^z$$

Since both spins in the initial state point up, only terms containing σ^- and σ^z can contribute:

$$S_{12}\left|{}^3S_1, m_j=+1\right\rangle=\sqrt{\frac{1}{4\pi}}\,3\left[\sin^2\theta\left(\sigma_1^-\sigma_2^-\,e^{2i\varphi}\right)+\cos^2\theta\,\sigma_1^z\sigma_2^z\right]\left|\uparrow\uparrow\right\rangle+$$

$$+\sqrt{\frac{1}{4\pi}}\,3\sin\theta\cos\theta\left[\left(\sigma_1^-\sigma_2^z+\sigma_1^z\sigma_2^-\right)e^{+i\varphi}\right]\left|\uparrow\uparrow\right\rangle-\sqrt{\frac{1}{4\pi}}\sigma_1^z\sigma_2^z\left|\uparrow\uparrow\right\rangle=$$

$$\sqrt{\frac{9}{4\pi}}\sin^2\theta\, e^{2i\varphi}\left|\downarrow\downarrow\right\rangle+\sqrt{\frac{1}{4\pi}}\left(3\cos^2-1\right)\left|\uparrow\uparrow\right\rangle+\sqrt{\frac{9}{4\pi}}\sin\theta\cos\theta\, e^{+i\varphi}\left|\downarrow\uparrow+\uparrow\downarrow\right\rangle$$

We compare this now to the wave function for the D-state, inserting the correct expressions for the spherical harmonics and Clebsch-Gordon coefficients:

$$\left|{}^3D_1, m_j=+1\right\rangle = \sqrt{\frac{3}{5}}Y_{22}\left|\downarrow\downarrow\right\rangle-\sqrt{\frac{3}{10}}Y_{21}\left|\frac{\uparrow\downarrow+\downarrow\uparrow}{\sqrt{2}}\right\rangle+\sqrt{\frac{1}{10}}Y_{20}\left|\uparrow\uparrow\right\rangle=$$

$$\sqrt{\frac{3}{5}}\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{i2\varphi}\left|\downarrow\downarrow\right\rangle+\sqrt{\frac{3}{10}}\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi}\left|\frac{\uparrow\downarrow+\downarrow\uparrow}{\sqrt{2}}\right\rangle+\sqrt{\frac{1}{10}}\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right)\left|\uparrow\uparrow\right\rangle$$

$$=\sqrt{\frac{9}{32\pi}}\sin^2\theta\, e^{i2\varphi}\left|\downarrow\downarrow\right\rangle+\sqrt{\frac{9}{32\pi}}\sin\theta\cos\theta\, e^{i\varphi}\left|\uparrow\downarrow+\downarrow\uparrow\right\rangle+\sqrt{\frac{1}{32\pi}}\left(3\cos^2\theta-1\right)\left|\uparrow\uparrow\right\rangle$$

Comparison shows exact agreement with exception of an overall normalization factor of 1/√8, q.e.d.