WIGNER FUNCTION

- In QM, we know that p and x are not compatible operators
  => we cannot measure x and p at the same time
- So how can we define a joint probability for (transverse) position and (transverse) momentum?
- One possible interpretation: Wigner Function W

$$W(\vec{r},\vec{p}) = \frac{1}{(2\pi\hbar)^3} \oiiint d^3\vec{r}\,' e^{i\vec{p}\cdot\vec{r}\,'/\hbar} \psi^*\left(\vec{r} - \tfrac{\vec{r}\,'}{2}\right) \psi\left(\vec{r} + \tfrac{\vec{r}\,'}{2}\right)$$

- Can rigorously calculate things like $\left\langle \frac{\mathbf{XP}_x + \mathbf{P}_x\mathbf{X}}{2} \right\rangle = \oiiint d^3\vec{r} \oiiint d^3\vec{p}\; x p_x W(\vec{r},\vec{p})$
- Disadvantage: not guaranteed to be positive; oscillatory
- Limited usefulness to address our question

TECHNICAL DETAILS OF THE WIGNER FUNCTION

$$W(\vec{r},\vec{p}) = \frac{1}{(2\pi\hbar)^3}\iiint_{\text{Phase Space}} d\vec{r}' e^{-i\vec{p}\cdot\vec{r}'\hbar}\psi^*\left(\vec{r}-\frac{\vec{r}'}{2}\right)\psi\left(\vec{r}+\frac{\vec{r}'}{2}\right)$$

lets work on it in just 1-dim (1 spatial coordinate and 1 momentum):

$$W(x,p) = \frac{1}{2\pi\hbar}\int dx' e^{-ipx'/\hbar}\psi^*\left(x-\frac{x'}{2}\right)\psi\left(x+\frac{x'}{2}\right).$$

What should this function satisfy if it is to be taken as a joint probability density?

- It should be real: $W = W^*$

This is already satisfied, for taking the complex conjugate gives

$$W(x,p) = \frac{1}{2\pi\hbar}\int dx' e^{ipx'/\hbar}\psi\left(x-\frac{x'}{2}\right)\psi^*\left(x+\frac{x'}{2}\right)$$

and it is always possible to take $x'' = -x'$, in which case we have $W = W^*$.

TECHNICAL DETAILS OF THE WIGNER FUNCTION II

Integrate now the Wigner function over all momenta

$$\int_{-\infty}^{\infty} dpW(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dpe^{-ipx'/\hbar}\psi\left(x-\frac{x'}{2}\right)\psi^*\left(x+\frac{x'}{2}\right)$$

$$\int_{-\infty}^{\infty} dpe^{-ipx'/\hbar} = 2\pi\hbar\delta(x')$$

$$\int_{-\infty}^{\infty} dpW(x,p) = \psi(x)\psi^*(x) = \left|\psi(x)\right|^2$$

Integrate now the Wigner function over all coordinates

$$\int_{-\infty}^{\infty} dxW(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dxe^{-ipx'/\hbar}\psi\left(x-\frac{x'}{2}\right)\psi^*\left(x+\frac{x'}{2}\right)$$

this is not straightforward since ψ and ψ^* also depend on x. Introduce, however, the following change of variables

$$x_1 = x + \frac{x'}{2}$$
$$x_2 = x - \frac{x'}{2}$$

the inverse transformation is then

$$x' = x_1 - x_2$$
$$x = \frac{1}{2}(x_1 + x_2)$$

and the Jacobian for this transformation is

$$\left|\frac{\partial(x,x')}{\partial(x_1,x_2)}\right| = \begin{vmatrix} \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial x_2} \\ \frac{\partial x'}{\partial x_1} & \frac{\partial x'}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{vmatrix} = 1$$

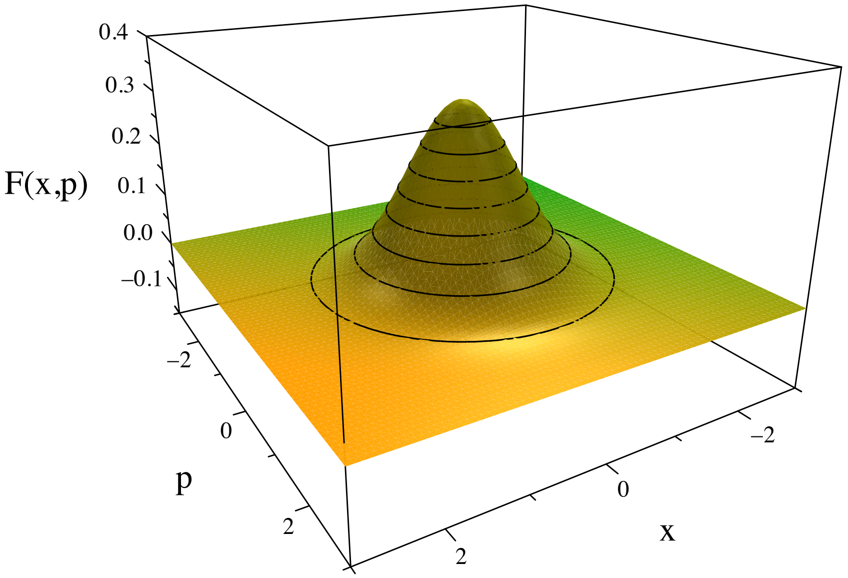
thus $dxdx' = dx_1dx_2$ and

$$\begin{aligned}
\int_{-\infty}^{\infty} dxW(x,p) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-ipx_1/\hbar}e^{ipx_2/\hbar}\psi^*(x_2)\psi(x_1) \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx_1 e^{-ipx_1/\hbar}\psi(x_1)\cdot\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx_2 e^{ipx_2/\hbar}\psi^*(x_2) \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx_1 e^{-ipx_1/\hbar}\psi(x_1)\cdot\left[\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx_2 e^{-ipx_2/\hbar}\psi(x_2)\right]^* \\
&= \tilde{\psi}(p)\tilde{\psi}^*(p) \\
&= \left|\tilde{\psi}(p)\right|^2
\end{aligned}$$

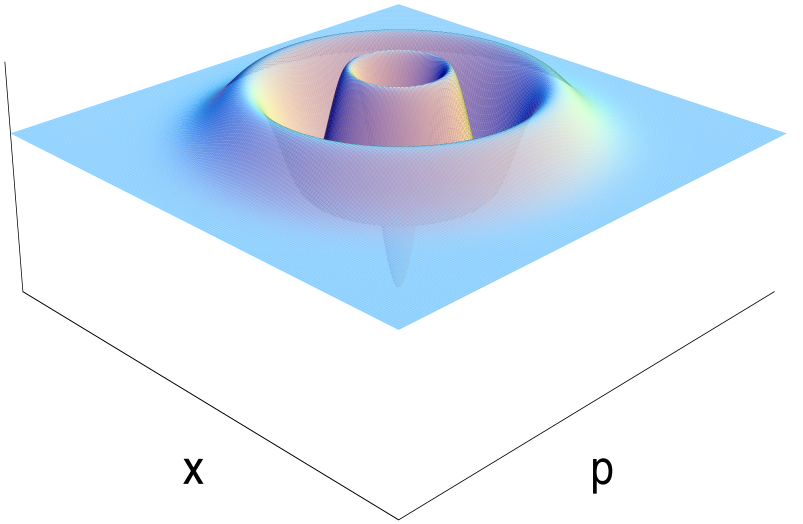
EXAMPLE: HARMONIC OSCILLATOR

- **Ground state:** $\psi(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$
  - No correlation between *x* and *p* at all.
  - Both are Gaussians with minimum width
  - $\langle x^2p^2\rangle = \langle x^2\rangle\langle p^2\rangle = ½·½\ \hbar^2$
- **Higher excited states (ψ_n)**
  - Both $\langle x^2\rangle$ and $\langle p^2\rangle$ increase by factor $2n + 1$
  - However, $\langle x^2p^2\rangle$ increases less so that correlation becomes negative:
    - For 1st excited state, r = -0.667
    - For 2nd excited state, r = -0.857
    - For 3rd excited state, r = -0.923…
  - Due to negative-going parts of Wigner Function

Wigner Function for HO ground state *)



Wigner Fnct for HO 3rd excited state

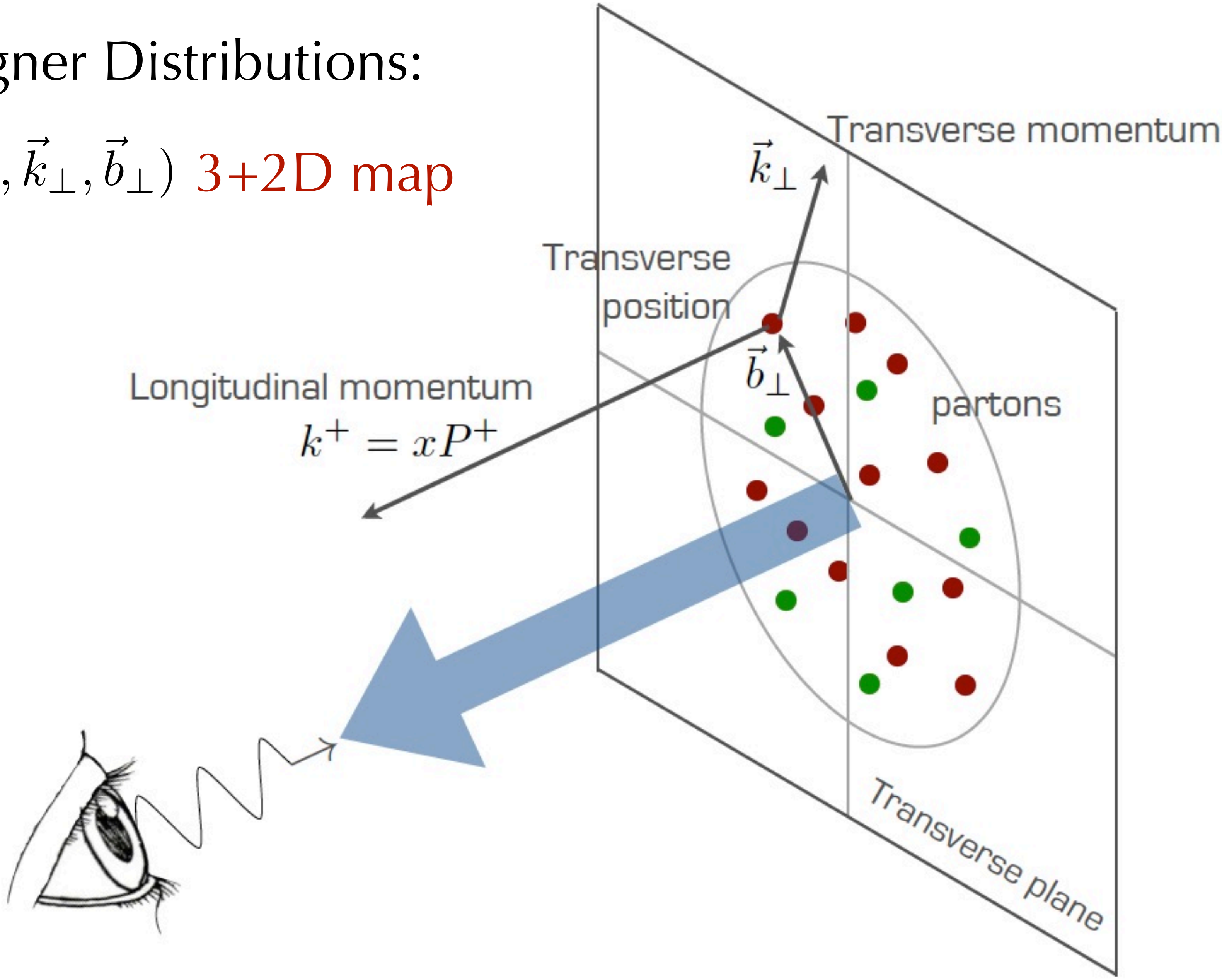


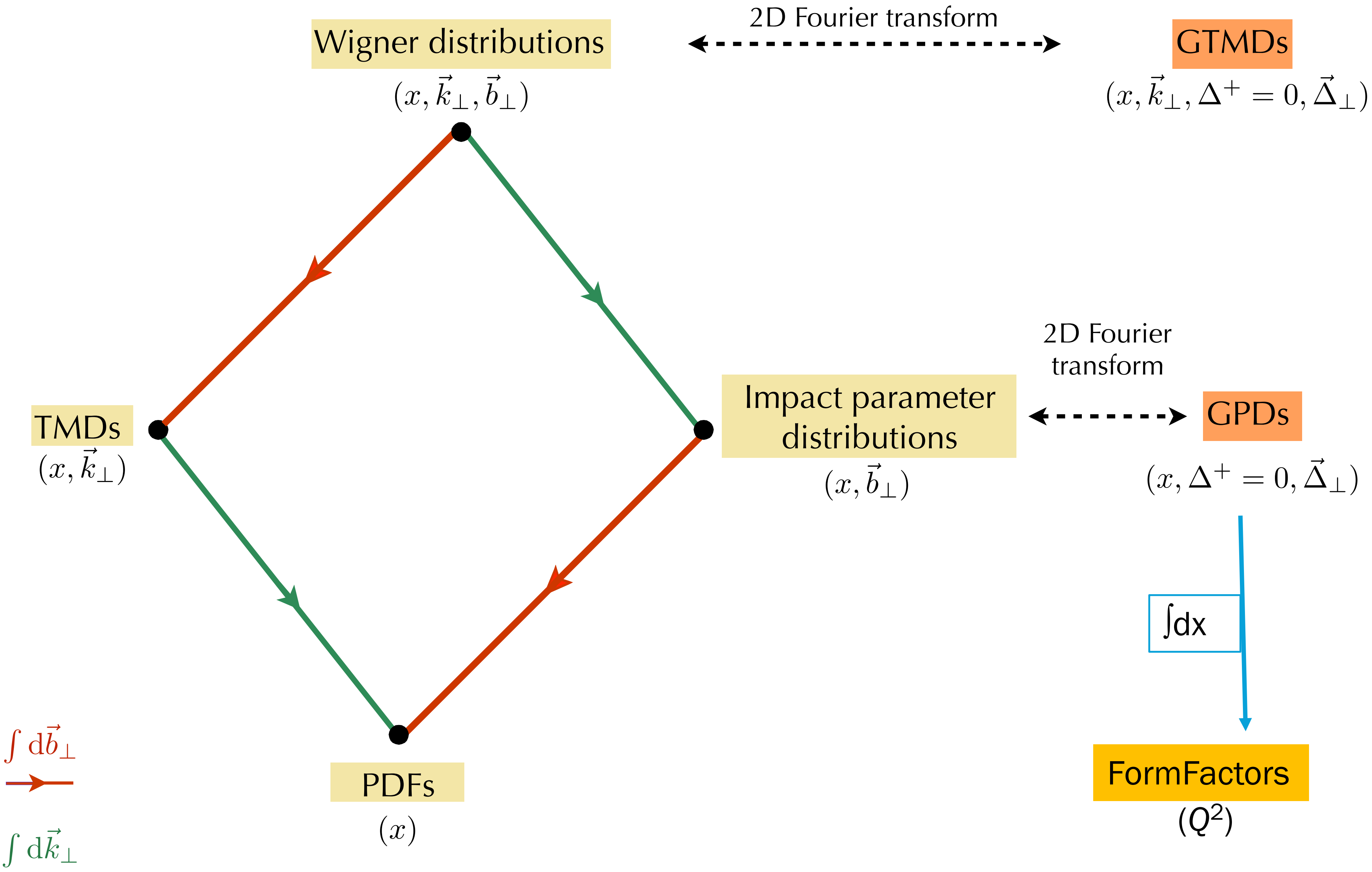
$$*)\ Result:\ \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{m\omega x^2}{\hbar}} \sqrt{\frac{1}{\pi\hbar m\omega}}\, e^{-\frac{p^2}{\hbar m\omega}}$$

BARBARA PASQUINI (UNIVERSITA DI PAVIA & INFN)

Wigner Distributions:

$\rho(x, \vec{k}_\perp, \vec{b}_\perp)$ 3+2D map





Wigner distributions

$(x, \vec{k}_\perp, \vec{b}_\perp)$

2D Fourier transform

GTMDs

$(x, \vec{k}_\perp, \Delta^+ = 0, \vec{\Delta}_\perp)$

TMDs

$(x, \vec{k}_\perp)$

Impact parameter distributions

$(x, \vec{b}_\perp)$

2D Fourier transform

GPDs

$(x, \Delta^+ = 0, \vec{\Delta}_\perp)$

∫dx

PDFs

(x)

FormFactors

(Q^2)

$\int d\vec{b}_\perp$

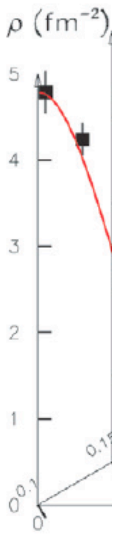
$\int d\vec{k}_\perp$

Lorcé, BP, Vanderhaeghen, JHEP05 (2011) 041

The unpolarized GPD H

$H^q(x,$

extrapolating
in the unmeasured
x-range



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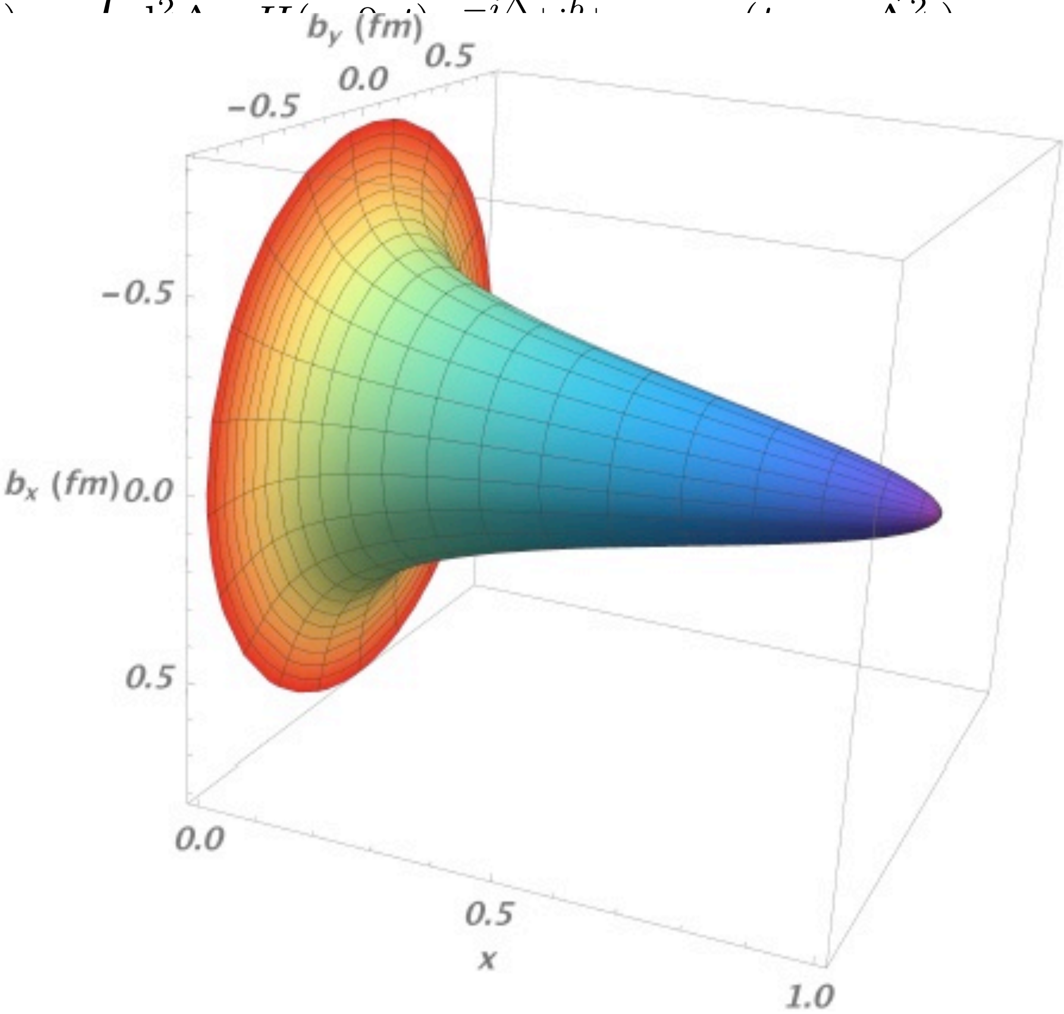
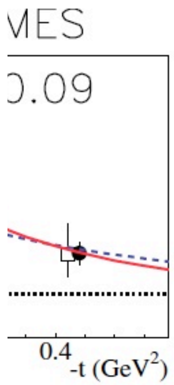




in t

$b_\perp$

Guidal et al., Rep. Prog. Phys. 76 (2013) 066202

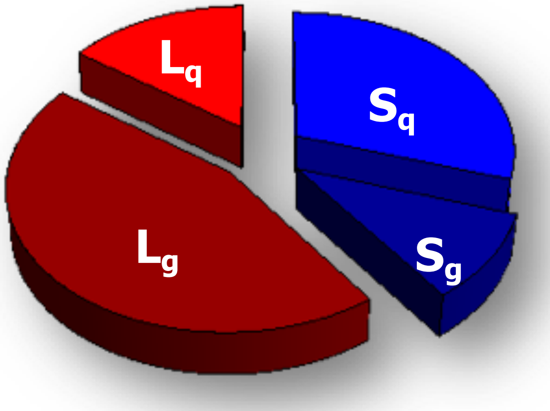
Angular Momentum Relation (Ji's Sum Rule)

X. Ji, PRL 78 (1997) 610

quark and gluon contribution to the nucleon spin

$$J^{q,g} = \frac{1}{2}\int_{-1}^{1} dx\, x\, \Big(\underset{\substack{\Big\downarrow \\ \text{at } \xi = 0 \text{ unpolarized PDF}}}{H^{q,g}(x,\xi,0)} + \underset{\substack{\Big\downarrow \\ \text{not directly accessible}}}{E^{q,g}(x,\xi,0)}\Big)$$

$$J^q = L^q + \overset{\substack{\frac{1}{2}\Delta\Sigma \text{ from DIS} \\ \uparrow}}{S^q}$$



$$J^g = L^g + \overset{\substack{\frac{1}{2}\Delta g \text{ from DIS} \\ \uparrow}}{S^g}$$

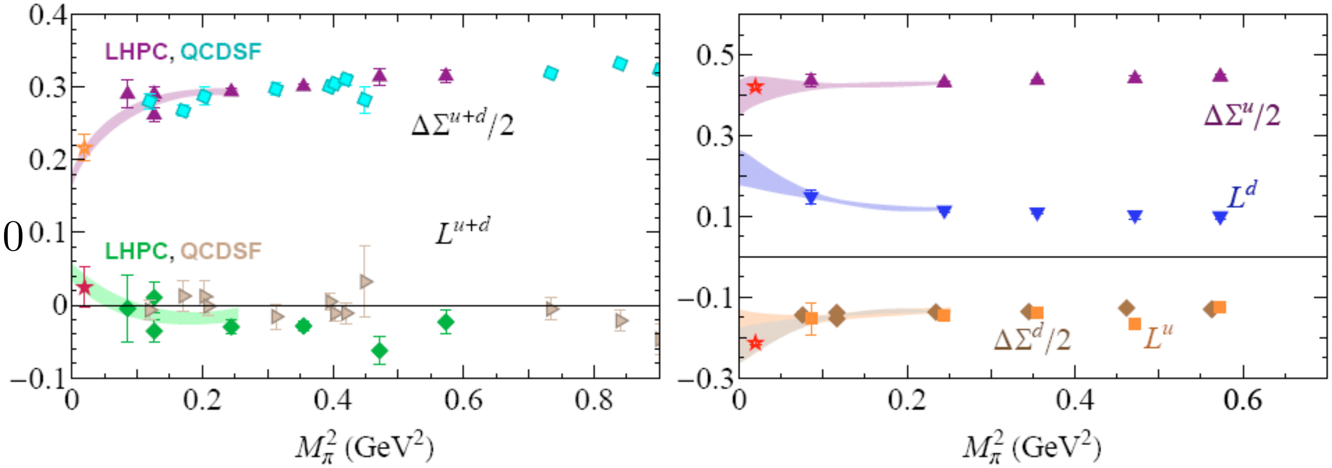
- Requires extrapolation at t=0
- Requires spanning x at fixed values of ξ ($\xi = 0$ is the most convenient)
- Does not have an interpretation as angular momentum density as a function of x

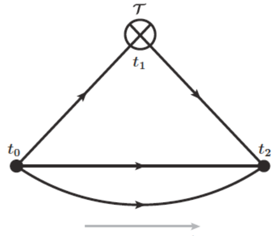
Lattice Calculations of Angular Momentum

without disconnected insertion

$L^u + L^d \approx 0$

$J^u > 0, J^d \approx 0$



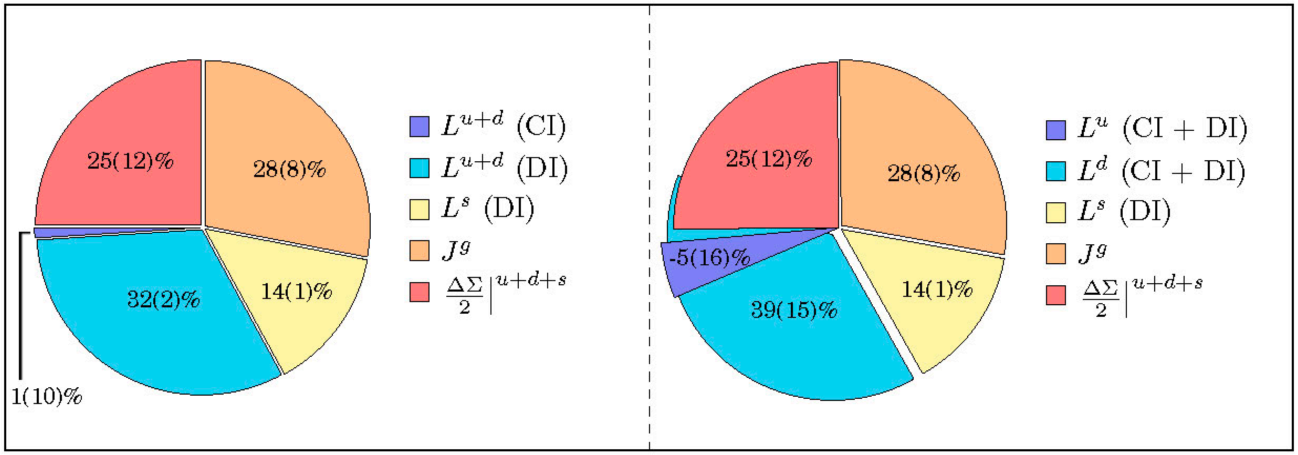


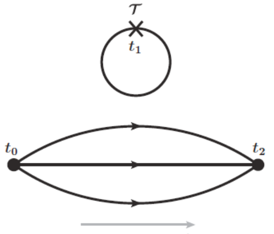
connected insertions (CI)

with disconnected insertion

$L^u + L^d \approx 33\%$

$(m_\pi = 330 \text{ MeV})$





disconnected insertions (DI)

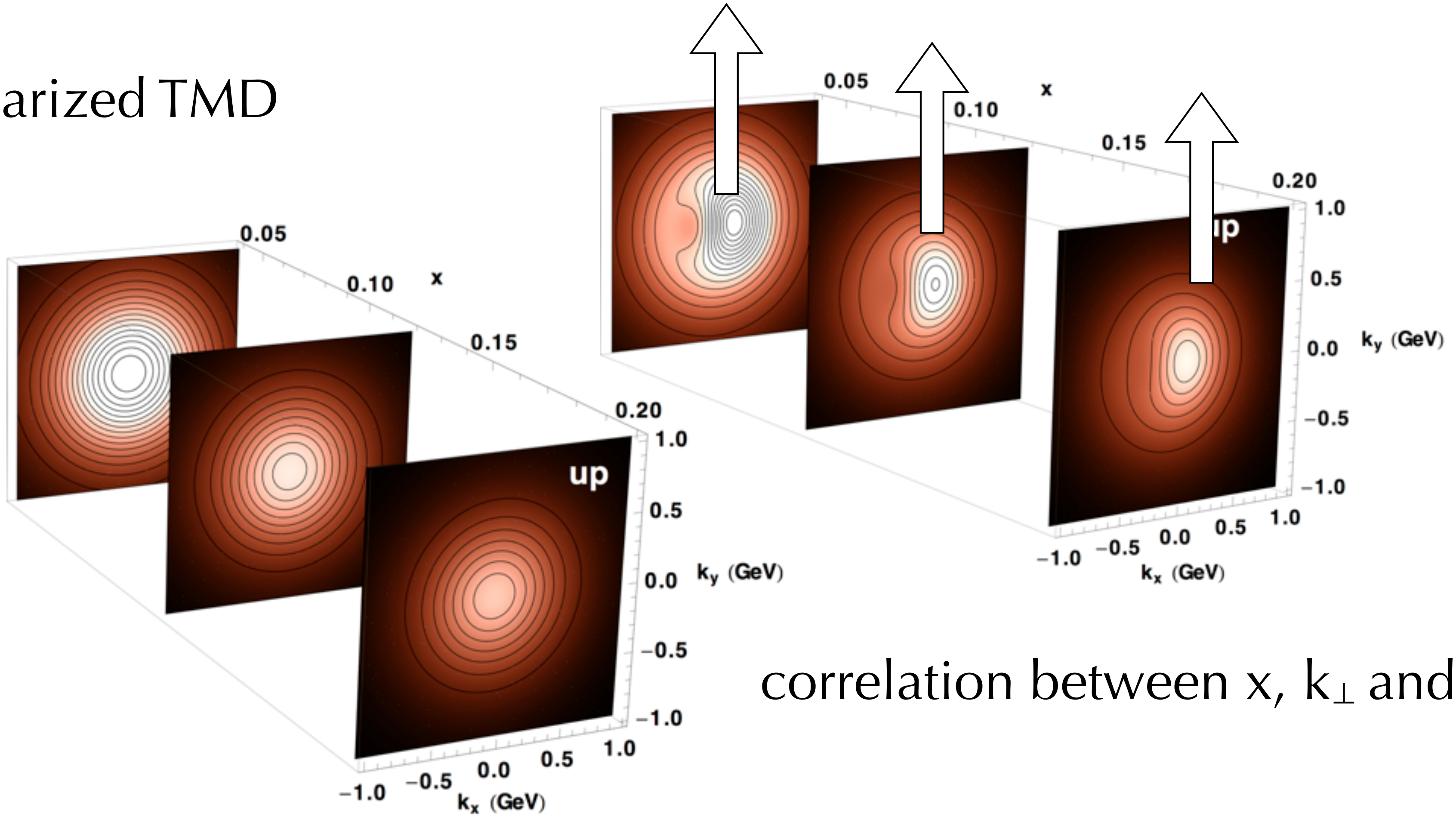
Deka et al., PRD ***91*** *(2015) 014505*

Consistent with recent lattice calculation at the physical pion mass
C. Alexandrou et al., PRL **119** (2017) 142002

MORE ON TMDS:

polarized quarks and/or polarized target

unpolarized TMD



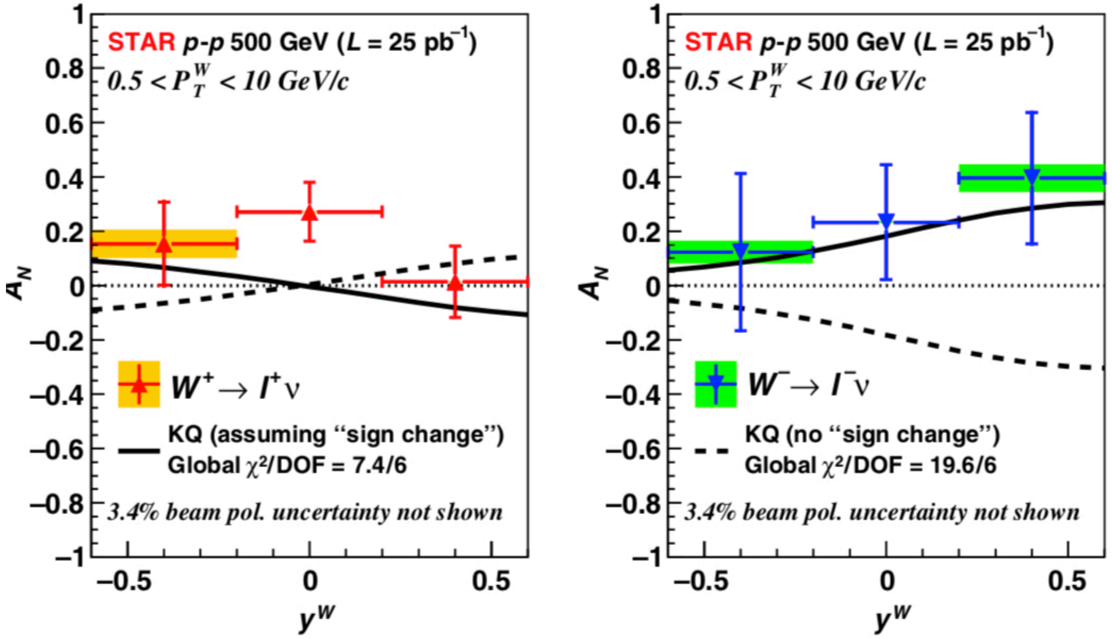
correlation between x, $k_\perp$ and spin

correlation between x and $k_\perp$

First hints of sign change

$p^{\uparrow}p \to W^{\pm}/Z$

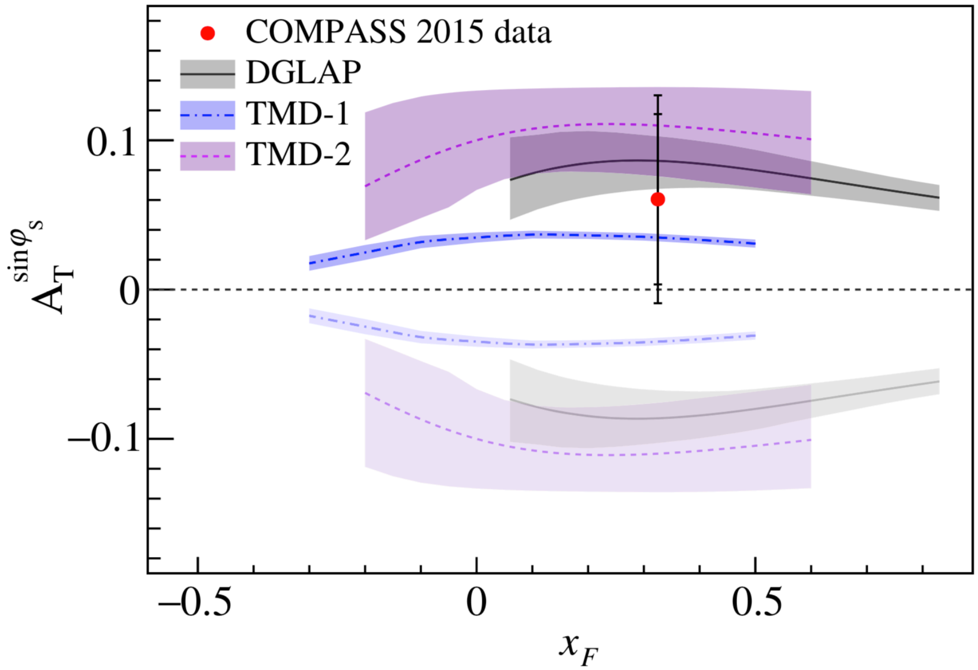
@ RHIC-STAR Coll.
PRL 116(2016)132301



sign change

no sign change

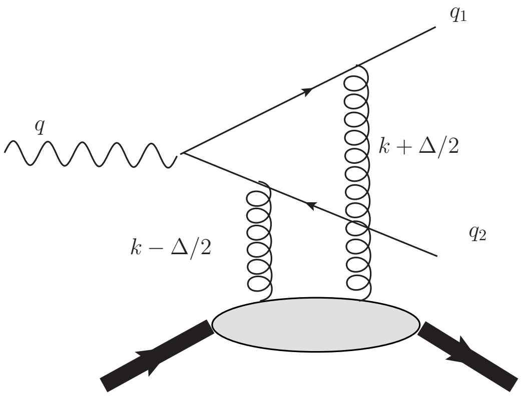
Drell-Yan $\pi p \to \mu\mu X$
@ COMPASS
PRL 119(2017)112002



sign change

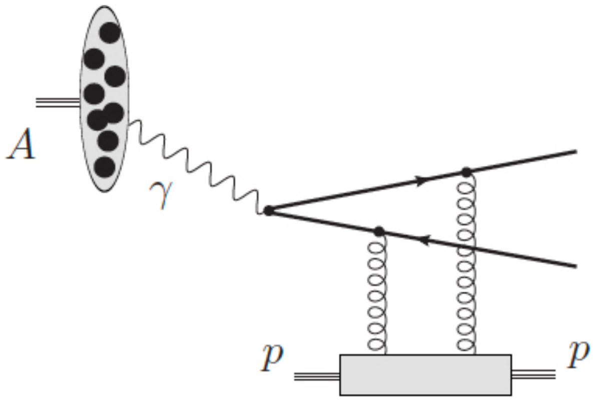
no sign change

Wigner Distributions (WD) and GTMDs from



Exclusive dijet production in ep DIS (gluon GTMDs)

Hatta, Xiao, Yuan, PRL 116 (2016) 202301
Hatta, Nakagawa, Xiao, Yuan, Zhao, PRD 95 (2017) 114032
Ji, Yuan, Zhao, PRL 118 (2017) 192004



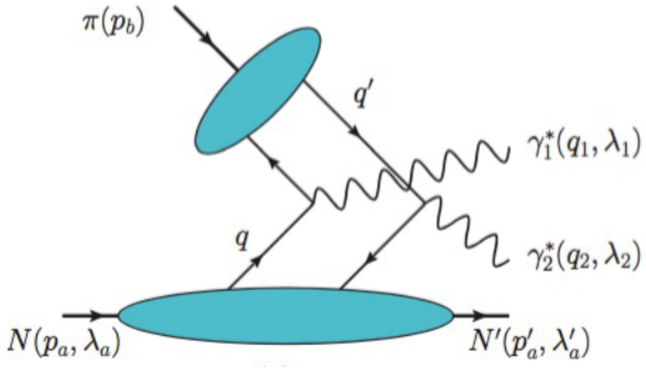
Exclusive dijet production in pA UPC (gluon GTMDs)

Hagiwara, Hatta, Pasechnik, Tasevsky, Teryaev, PRD 96 (2016) 034009

Exclusive double quarkonia production in nucleon-nucleon collisions (gluon GTMDs)

Bhattacharya, Metz, Ojha, Tsai, Zhou, arXiv:1802.10550

Boussarie, Hatta, Xiao, Yuan, arXiv: 1807.08697



Exclusive pion-nucleon double Drell-Yan (quark GTMDs)

Bhattacharya, Metz, Zhou, PLB 771 (2017) 396