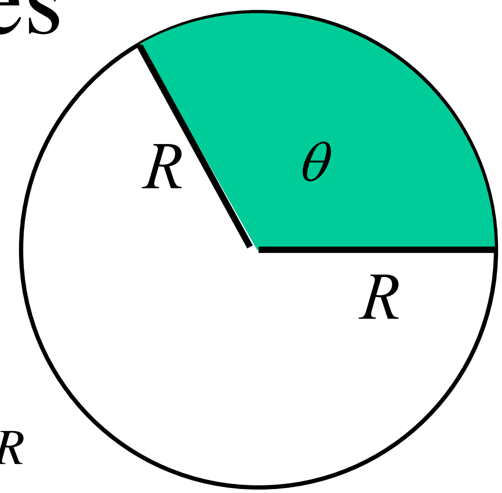
Motion in a Circle

- **So far:**
  Described **linear** motion of a mass point using **x**, **v**, **a**, *m*, **p**, **F**.
  Equations of motion: $\mathbf{a} = \Sigma\mathbf{F}/m$ or $\Delta\mathbf{p} = \Sigma\mathbf{F}\cdot\Delta t$; Equilibrium: $\Sigma\mathbf{F} = 0$
  Kinetic energy $K.E. = {}^1/_2\, mv^2$; $E_{tot} = K.E. + U_{pot}$; total energy and momentum conserved in the absence of external forces.
  Total energy change by external force **F**: $\Delta E_{tot} = \Delta W = \mathbf{F}\cdot\Delta\mathbf{s}$
- **Now:**
  We will study circular (rotating) motion. We will use a new set of variables to describe this motion:
  θ , ω , $\mathbf{L}$, $\boldsymbol{\tau}$
  and express equations of motion and K.E. in terms of these quantities.
- **Afterwards** , we will apply these ideas to rigid bodies rotating around a fixed axis. Will find new conserved quantities and new condition for equilibrium

"New" kinematic variables



Particle going around the origin on a circle of radius R

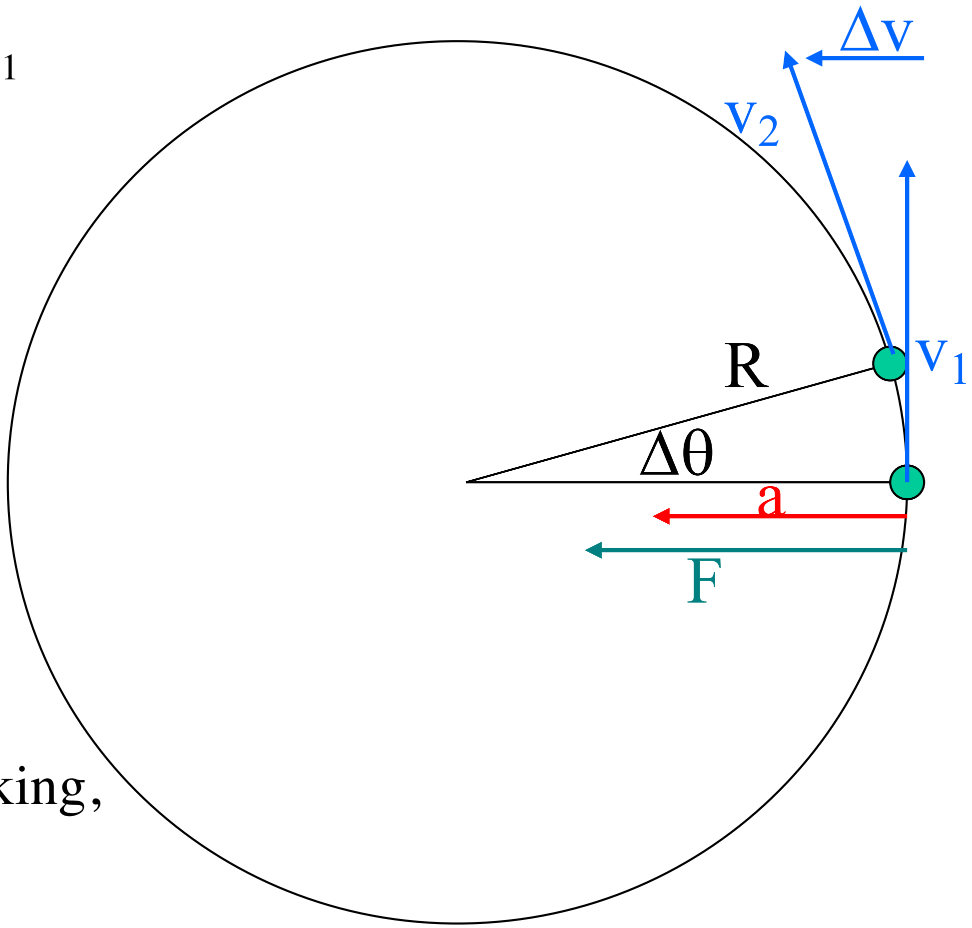
- Use angle θ to describe position:
  - Can be measured in degrees [º]
  - 360º is full circle
  - Circumference = distance once around the full circle = $2\pi R$
  - θ [º] /360 $\cdot$ $2\pi R$ tells us by how much distance (in m) the particle has moved around the perimeter
  - => Can also express θ in radians [rad]: θ [rad] = $2\pi \cdot \theta$ [º] /360
  - Distance traveled around perimeter = $R \cdot \theta$ in radians
- Angular velocity describes how fast particle goes around the circle:
  - rps = revolutions per second (1rps = 60 RPM - "rounds per minute")
  - If it takes time T to go all the way around once, then $1/T$ = number of rps
  - After some time Δt, particle has moved by $\Delta\theta$ [degrees] = $360° \cdot rps \cdot \Delta t$; $\Delta\theta$ [radians] = $2\pi \cdot rps \cdot \Delta t = \omega \Delta t$; $\omega = 2\pi \cdot rps$ = angular velocity
- Linear speed $|\mathbf{v}| = 2\pi R/T = rps \cdot 2\pi R = \omega R$
  - The higher the angular velocity, the higher the linear speed
  - The further away from the center (the larger R), the higher the linear speed

Why…

- …do we introduce new variables?
  - Simplify description: need only **one** number for position (and **one** number for velocity); otherwise need more numbers since position, velocity are 2- or 3-dimensional vectors!
  - Can apply what we learn to rotation of extended objects (spinning wheels, cylinders, fans, blades, tops,…)
  - Will discover new conservation law (important for astronomy, ice skaters, all other rotating objects, fundamental laws of Physics): Conservation of angular momentum $\mathbf{L}$
  - Study new conditions for equilibrium (net torque = 0).

Something special about circular motion…
… it requires a (centripetal) force!
(even if you aren't speeding up or slowing down)

- After a short time Δt: $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$
- Change larger if $\mathbf{v}_1$, $\mathbf{v}_2$ larger
- Time Δt shorter if ω larger
  - $\omega = 2\pi/T$, $T = 2\pi R/v$ =>
- $\mathbf{a}_c = \Delta\mathbf{v}/\Delta t = v^2/R = \omega^2 R$
  RADIALLY inwards
  ("centripetal" acceleration)
- $\mathbf{F}_c = m\mathbf{a}_c$
  (centripal force)
- Examples:
  car driving around a corner, banking, ball on string, space station…



Angular Momentum **L**

- **L** = $mR^2\,\omega = mRv$ = **angular momentum**
- …is another **conserved quantity** in Physics if no **tangential** force is acting:
  - if R = const. this follows from conservation of (kinetic) energy: K.E. = const. $\Rightarrow v$ = const. $\Rightarrow mRv = L$ = const.
  - if radius R **de**creases: radial force does **positive** work $\Rightarrow v$ **in**creases $\Rightarrow mRv = L$ = const.
  - if radius R **in**creases : radial force does **negative** work $\Rightarrow v$ **de**creases $\Rightarrow mRv = L$ = const.
- … points in the direction of axis of rotation (right hand rule:
  + = counter-clockwise rotation, - = clockwise rotation)
- Example: Ball at the end of a string: how do I, E, L, v vary with R, ω?
- Extremely useful and important (just like conservation of **p** and E) - see later examples with rotating objects

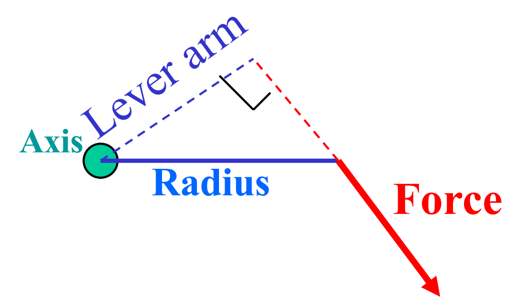
Now: extend to rotation of an extended object around a fixed axis

- **So far:**
  We studied the motion of a **single** object (mass point) on a circle around the origin. Motion described by:
  θ , ω , $\mathbf{L}$, …
- **Now:** We will apply these ideas to rigid bodies rotating around a fixed axis.
- Consider extended object as a collection of (very many) mass points (atoms), each moving on a circle of radius r_P (= distance from axis).
- Obviously, each mass point has different velocity, acceleration, forces acting on it…
  But: all have the **same** ω. All have the same angle θ up to a constant offset. The whole object can be described by a single θ, ω and a single L.

Moment of Inertia

- Can write $L = I\ \omega$
- Plays similar role in rotational motion as mass (inertia) plays in linear motion
- Since L is proportional to m, R^2 and ω, I must be proportional to R^2 and m – in fact, it's the mean r^2 of all mass in an object, times its total mass M.
- Examples:
  – Skinny objects rotating around their long axis have small I, extended objects or long objects rotating around their short axes have large I.
  – Objects of same shape but higher mass have higher moment of inertia.
  – Objects of same overall size and mass have larger I if the mass is concentrated far away from axis (Disk race)
- Conservation of angular momentum **L**:
  – If I increases, ω must decrease (moving mass outwards)
  – If I decreases, ω must increase (moving mass inwards)
  – Examples: ballerina, figure skating, rotating chair + person with dumbbells

Finally… - Torque!



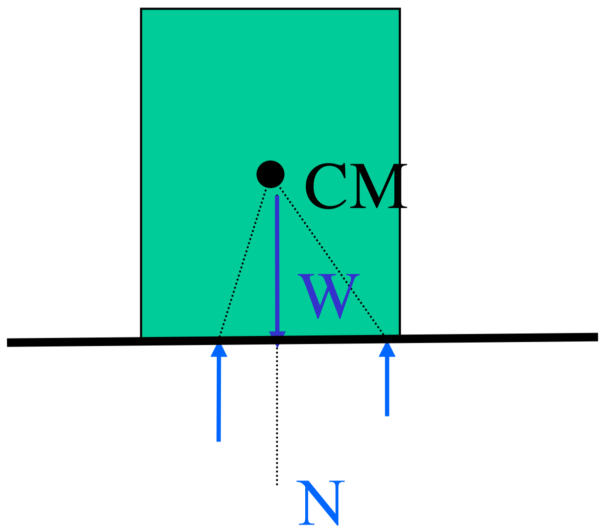
- Tangential force times leverarm
- Plays the role of force in linear motion
- $\tau = F{\cdot}l$:
  proportional to the force exerted
  proportional to the length of the lever arm
  only the part of the radius vector **perpendicular** to the force counts!
  $\tau = F{\cdot}l \sin(\theta) = \boldsymbol{F} \times \boldsymbol{l}$
- Unbalanced torque will speed up rotational motion:
  Change in angular velocity $\Delta\omega = \tau / I \, \Delta t$
- Unbalanced torque is the cause for any change in angular momentum:
  $\tau = \Delta L \,/\, \Delta t$

New Requirements for Static Equilibrium

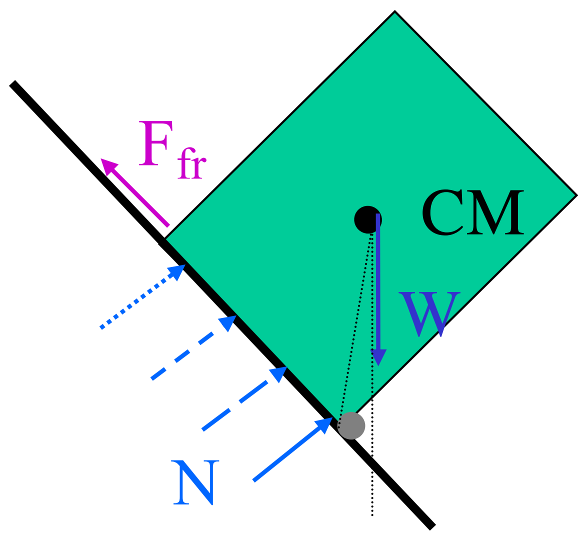
- **So far:**
  Mass points: Require $\Sigma\, \mathbf{F}_i = 0$ for static equilibrium (otherwise $\mathbf{a} \neq 0$). Include weight, normal forces, friction, tension in attached strings, other external forces.
- **Now:**
  Extended objects: **Still** require $\Sigma\, \mathbf{F}_i = 0$. **But** : not sufficient -> if forces act on different parts of object, net torque could be non-zero => rotation.
  Therefore : Require $\Sigma\, \boldsymbol{\tau}_i = 0$ as well.

Example I

- Center of gravity must be straight above supporting area

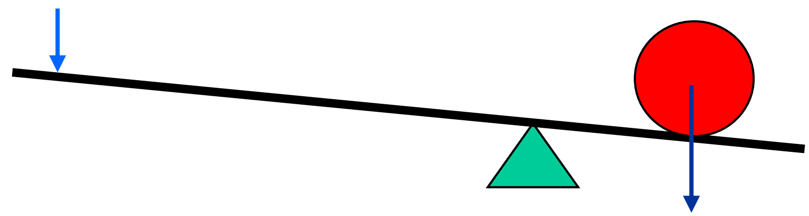


- Tipping over

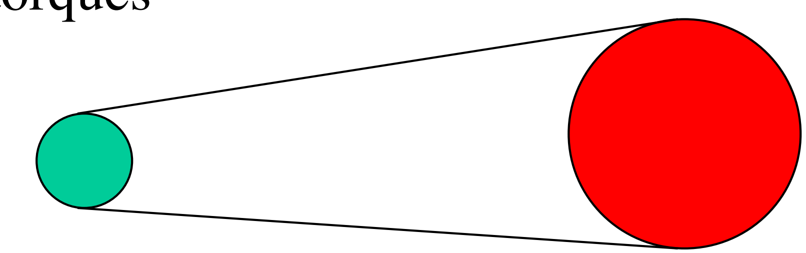


Levers and gears

- Levers: small force times large leverarm = large force times short leverarm.
  - Net torque = 0
    (const. ang. velocity)
  - $l\,F = l\,F$
  - Same work done by either end.



- Chains and gears
  - Same Tension/force on either sprocket
  - Different leverarms -> different torques
  - Same work done: $\tau\,\Delta\Theta = T\,\Delta\theta$



Comparison linear motion with angular motion

- Position: $x(t)$
- Velocity: v
- Acceleration: a
- Mass: m
- Linear momentum: $p = mv$
- Force: F
- Newton's Law:
  $F = ma = \Delta p / \Delta t$
- K.E. = $m/2\ v^2$
- Momentum conserved if $\Sigma F = 0$
  (no net force)
- Change of K.E.:
  Δ K.E. = $\Delta W = F\ \Delta x$

- Angular Position: θ
- Angular velocity: rps, ω
- Angular acceleration: α
- Moment of Inertia: $I = \langle mR^2 \rangle$
- Angular Momentum: $L = I\ \omega$
- Torque: $\tau = R\, F_{\text{tan}}$
- "Newton's" Law:
  $\tau = I\alpha = \Delta L / \Delta t$
- K.E. = $I/2\ \omega^2$
- L conserved always if $\Sigma \tau = 0$
  (no net torque)
- Change of K.E.:
  $\Delta W = \tau\, \Delta\theta$

Summary: Motion is in 2D, but can be described by single (1D) variables