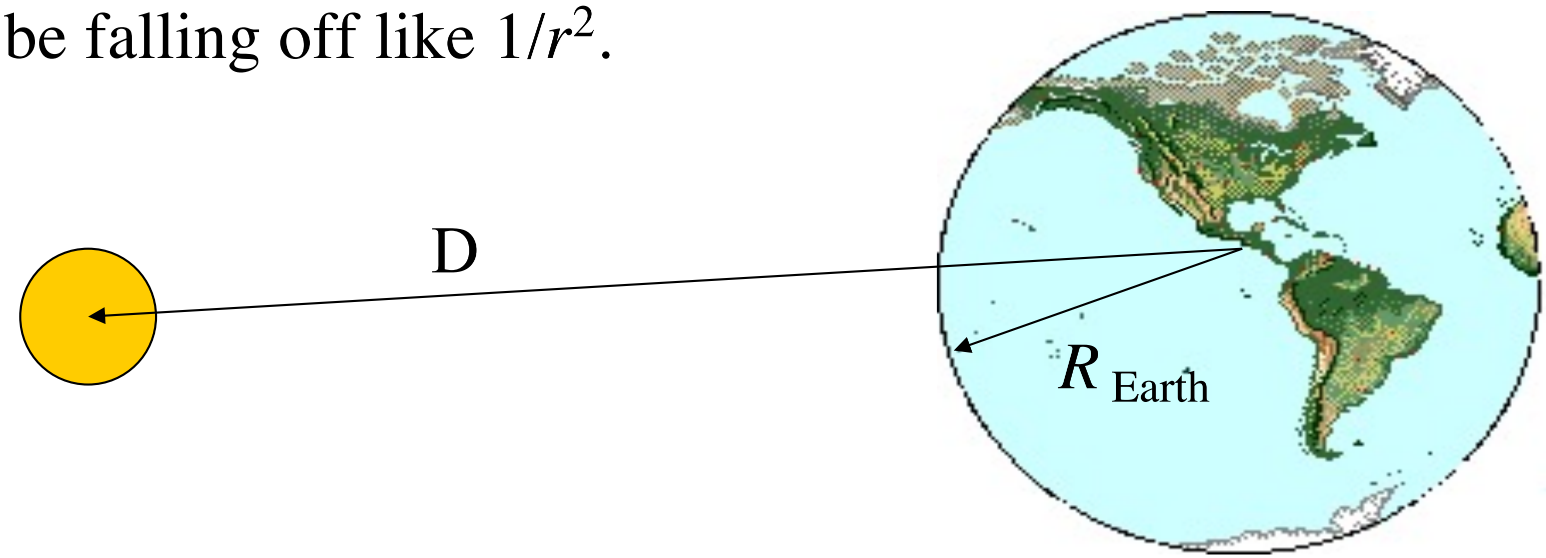
Gravitation

sez:

- Objects on the surface of Earth fall down with acceleration $a_{\text{rad}} = g = 9.81\ \text{m/s}^2$; Earth's radius is $R_{\text{Earth}} = 6380$ km.
- Moon circles the Earth once every 27.3 d = $2.36 \cdot 10^6$ s => $\omega = 2.66 \cdot 10^{-6}$/s . Moon is $D = 384{,}000$ km away => $a_{\text{rad}} = D\omega^2 = 0.00272\ \text{m/s}^2$ (3600 times smaller). D is 60 times bigger than R_{Earth} ! => Gravitational force must be falling off like $1/r^2$.



Newton's Law of Gravitation

- All masses m are accelerated with the same acceleration at the same distance from Earth => $F \propto m$ (since $a = F/m$)
- Logic: If Earth (mass M) attracts mass m with force proportional to m then that mass must attract Earth with a force proportional to M
- Newton's 3rd law => Earth's attraction on m must ALSO be proportional to M => $F \propto m\, M$
- The force is proportional to the distance squared: $F \propto 1/r^2$
- Need proportionality constant: G =>
  $F = G\, m\, M / r^2$
- Universal constant => Universal force law for any two bodies with masses m , M at distance r !
- Measure G using torsion balance => $G = 6.7 \cdot 10^{-11}$ N m^2/kg^2
- Plug in numbers: on the surface of earth,
  $g = F/m = G\, M / R_{\mathrm{Earth}}{}^2$ => $M = 5.97 \cdot 10^{24}$ kg

Important points

$$\vec{F}(\text{on } m_2 \text{ at } \vec{r}_2 \text{ due to } m_1 \text{ at } \vec{r}_1) = -G\frac{m_1 m_2}{\left|\vec{r}_2 - \vec{r}_1\right|^2} \text{ pointing from } m_2 \text{ to } m_1$$

- Universal $1/r^2$ force law - describes not only gravity, but also electromagnetism …
- Valid not only for point masses, but for spherical extended masses as well (measure Δr from the center)
- G can be measured with torsion balance (but it's hard because it is so small) -> Value: $6.673 \cdot 10^{-11}$N m^2/kg^2
- Force always acts **on** center of gravity of an object (roughly equal to center of mass)
- Force always points **to** center of gravity of attracting mass (along distance vector)
- Mass m exerts equal and opposite force on mass M as that exerted by M on m .
- Superposition: Gravitational forces add

Example I

- Gravity on the surface of moon
  - Radius of Moon: 1740 km = 27.3% of Earth's radius (roughly 1/4)
  - IF density were the same: Mass of Moon ≈ 1/64 Mass of Earth (really 1/81 = 1.23%)
  - Gravitational acceleration on Moon ≈ $(1/80)/(1/4)^2 = 1/5 = 2\ \mathrm{m/s^2}$ (really $1.62\ \mathrm{m/s^2}$)

Example II

- Project "Lagrange Point"
  Put a satellite between Sun and Earth where the net gravitational force is zero
  - Satellite can be stationary (looking at Earth)
- Where to put?
  - $M_{\text{Earth}} = 6{\cdot}10^{24}$ kg , $M_{\text{Sun}} = 2{\cdot}10^{30}$ kg , $D_{\text{E-S}} = 1.5{\cdot}10^{11}$ m .
  - Require $GmM_{\text{Earth}}/r^2 = GmM_{\text{Sun}}/(D_{\text{E-S}} - r)^2$
    $\Rightarrow (D_{\text{E-S}} - r) = 577\, r \Rightarrow r = D_{\text{E-S}}/578 = 259{,}000$ km.

Example III

- Two steel balls floating in space (initially at rest). Masses $M = 10$ kg, $m = 5$ kg, $d = 0.1$ m apart.
  - Initial gravitational attraction: $F = G\,m\,M/d^2 = 3.34{\cdot}10^{-7}$ N (on each)
  - Initial acceleration: $3.34{\cdot}10^{-8}$ m/s^2 for M, $6.68{\cdot}10^{-8}$ m/s^2 for m.
  - After 60 s, first one has moved 0.06 mm. Second has moved 0.12 mm.
  - Center of mass remains at same point (1/3 of the way from first to second mass).

Tidal forces

- Because gravitational forces vary like $1/r^2$, they can be different on different parts of the same object.
- Example: Gravitational force exerted by moon on Earth: Largest at point closest to moon, less large on center of Earth, least on opposite side.
- Variation of force over an object of diameter D: $\Delta F/m = \mathrm{D} \cdot \mathbf{2} GM/r^{\mathbf{3}}$
- Tend to elongate objects along connecting line
- Examples: Oceans on Earth (and atmosphere, Earth itself); Moon; black holes

Putting it all together…

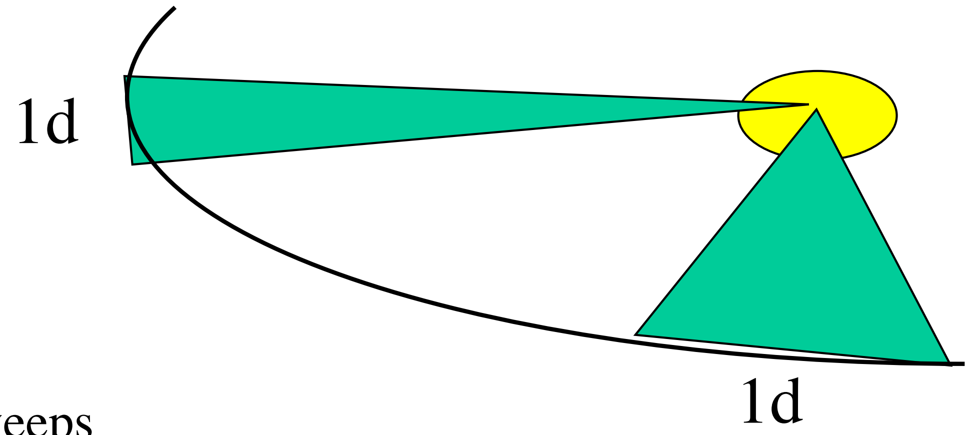
- Newton's 3 laws: a = F/m, action = reaction
- Linear motion: position, velocity, acceleration, momentum
- Circular motion: angular velocity, angular momentum, centripetal acceleration
- Energy (kinetic, potential,…)
- OUR FIRST REAL FORCE LAW: $F = G\, m\, M \,/\, r^2$

- => 3D motion in a gravitational field

First case: Satellite Motion

- What happens if you throw horizontally, faster and faster?
  - Earth is “curving away underneath flightpath”
  - Direction of gravitational force changes over time
- => if speed is fast enough, you get circular motion and no impact at all => Satellite!
- Minimum required speed at surface of Earth:
  $g = v^2/R$ => $v = \sqrt{gR} = \sqrt{9.8\,\mathrm{m/s^2} \times 6380{,}000\,\mathrm{m}} = 7900\,\mathrm{m/s}$
- Acceleration is 3600 times smaller where the moon is -> need only 1/60 of the speed to stay aloft!
- Geostationary satellites: $v/2\pi R = 1/86400\mathrm{s}$

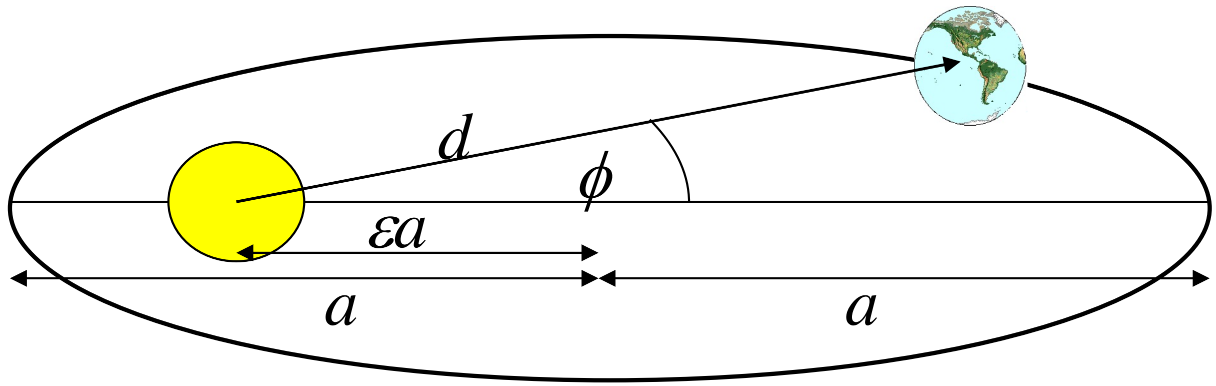
More general: Motion of planets and satellites



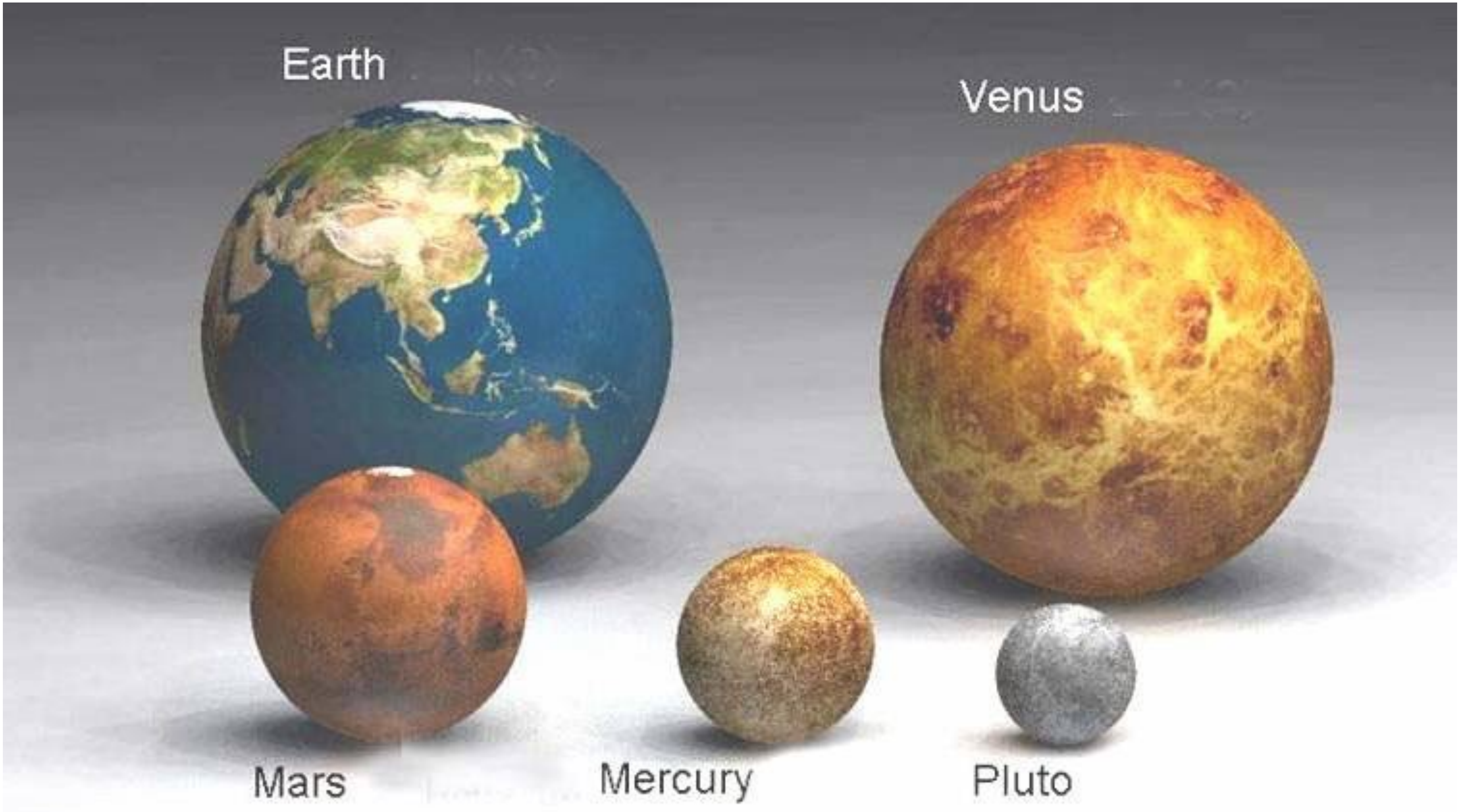
- Kepler's Laws:
  – Orbits of satellites (moons, planets,...) are elliptical
  – A line from the gravitating body (sun, planet,...) to the satellite sweeps out equal areas in equal times. (Conservation of angular momentum)
  – The square of the period T of an orbit is proportional to a^3 (a = major half axis of ellipse) and $1/M$.
- Circular orbit at distance R from central object with constant angular velocity ω
  – Centripetal force: $F_{\text{rad}} = m\,\mathrm{v}^2/R = m\,(2\pi R/T)^2/R = mR\,4\pi^2/T^2$
  – Gravitational force: $F_{\text{grav}} = GmM\,/R^2$
  – => $\omega^2 = 4\pi^2/T^2 = G\,M/R^3$
    => $T^2 = 4\pi^2\,(R^3\,/GM\,)$
  – Example: Satellite vs. Moon - Example: Miniature solar system

Motion of planets and Satellites (cont'd)

- Elliptical orbit



  - Gravitating body (sun, planet,…) located in one focal point of ellipse
  - Total " length" of ellipse is 2 a ("major half axis")
  - "Eccentricity" ε = distance from focal point to center/a
  - Major half axis a replaces R in 3rd Law:
    $T \propto a^{3/2}$
  - Conservation of angular momentum L requires $\omega \propto 1/d^2$ => "areas" law
  - => Kepler's Laws derived from Newton's Laws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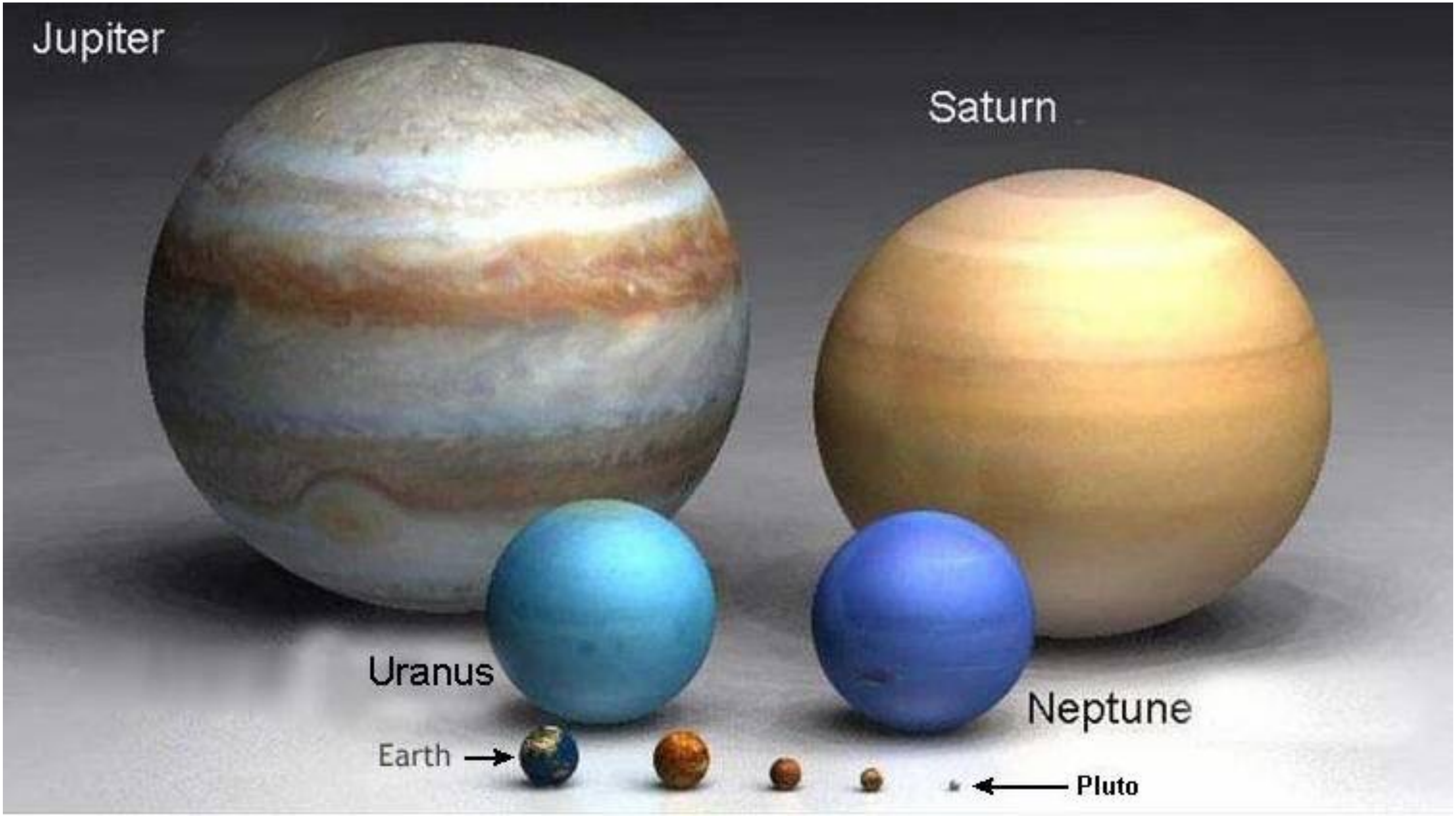
Venus
Mars
Mercury
Pluto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Earth
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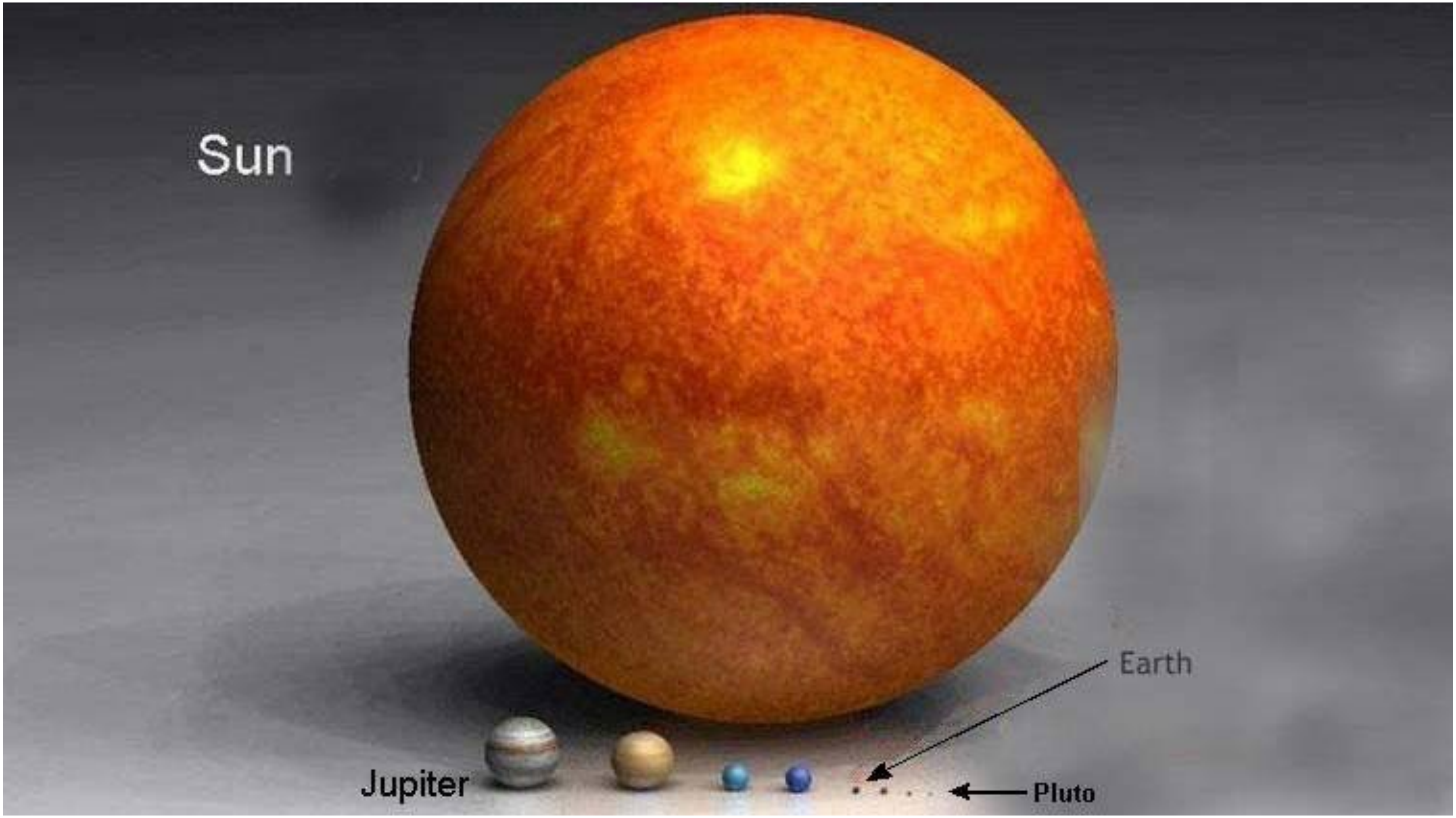
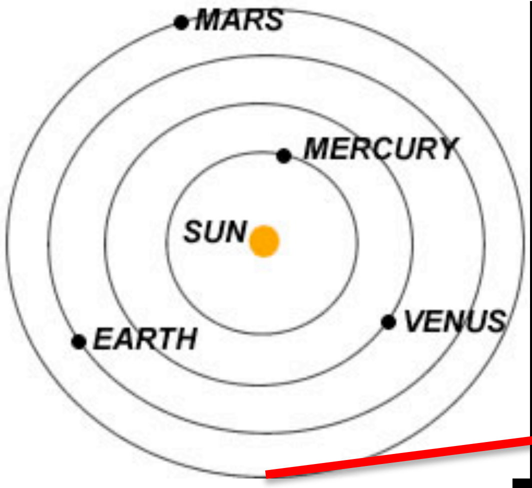
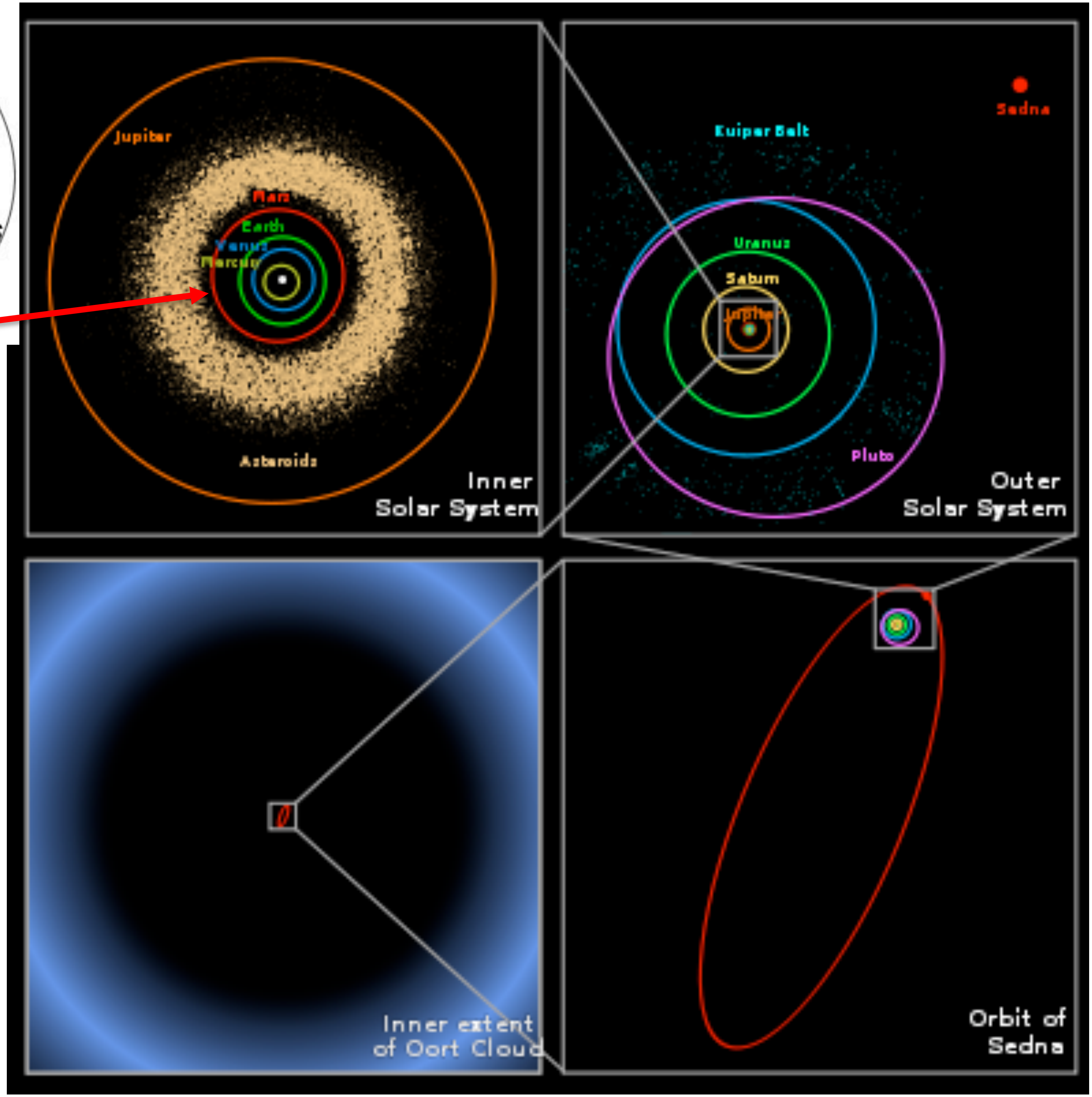
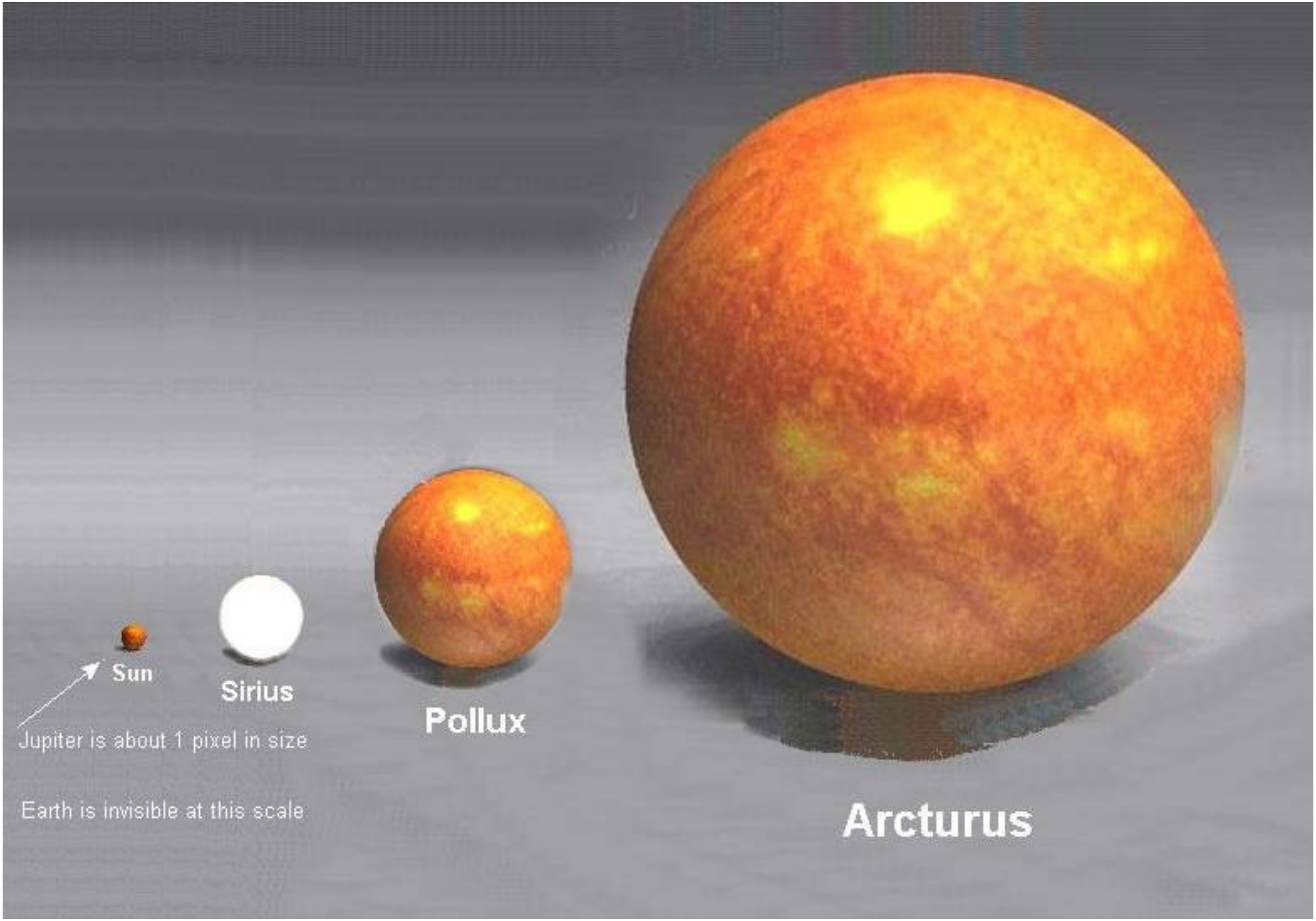
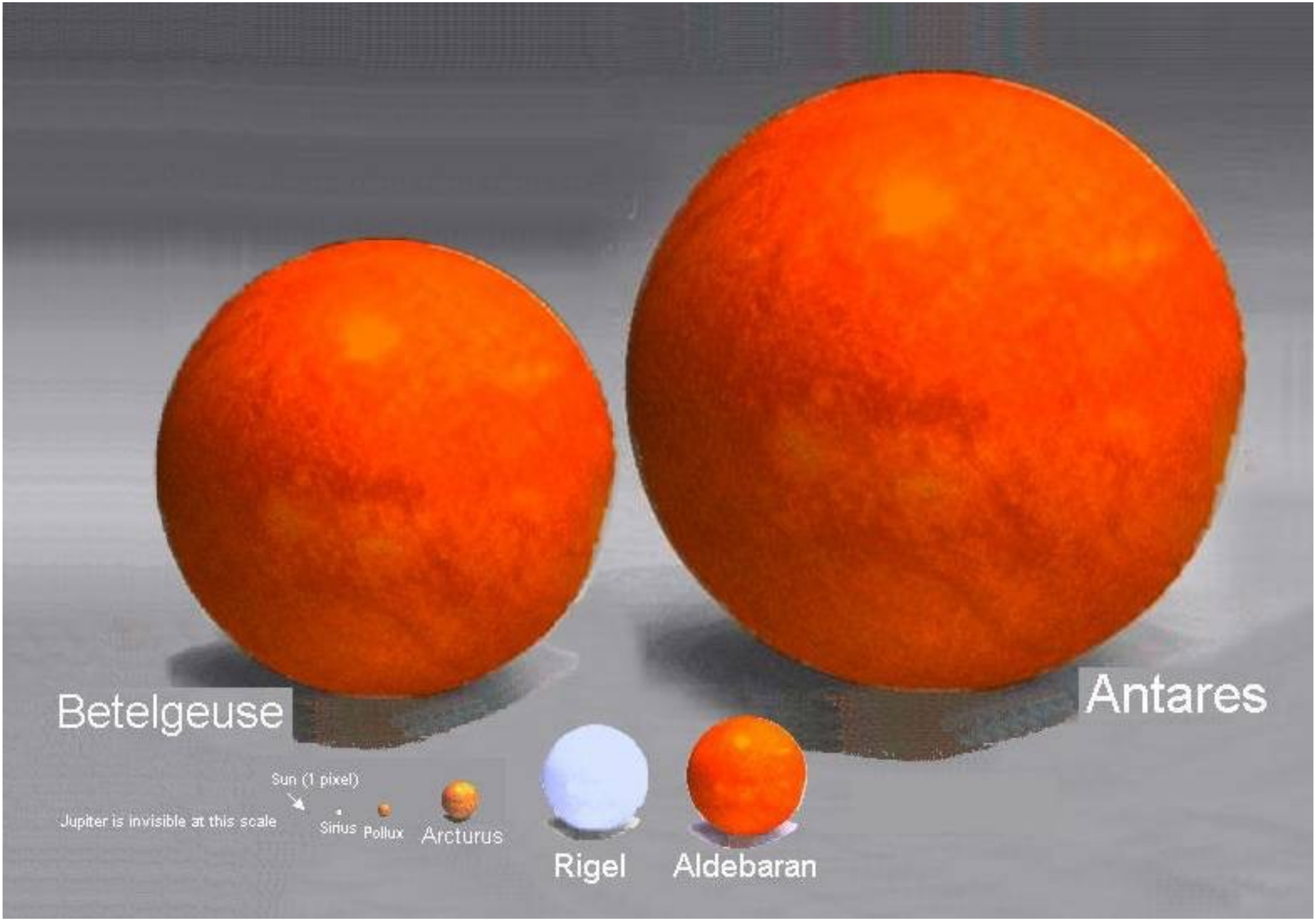

Sun
Earth
Jupiter
Pluto



The Solar System




Sun
Sirius
Pollux
Jupiter is about 1 pixel in size
Earth is invisible at this scale
Arcturus


Betelgeuse
Antares
Sun (1 pixel)
Jupiter is invisible at this scale
Sirius
Pollux
Arcturus
Rigel
Aldebaran

Gravitational Potential Energy

- Potential energy change = -Work done by gravity
- Close to Earth: $F = mg \Rightarrow W = -mg\Delta h \Rightarrow U = mgh$
- Moving further away, force is not constant but decreases, so energy increases less slowly: $\Delta U = GmM / r^2 \Delta r$
- Note: only motion away from or towards source of gravitational attraction changes potential energy
- General form for 2 masses m, M at a distance D:
  $U_{\text{grav}} = -GmM / D$
- Note: always negative! (We chose the reference point where $U = 0$ infinitely far away in this case)
- Note: It takes a FINITE amount of energy to escape Earth (the sun, moon, …) FOREVER!

Examples

- Two steel balls from before:
  (M = 10 kg, m = 5 kg, d = 0.1 m apart)
  - Initially $U_{grav} = -GmM/D = -3.34 \cdot 10^{-8}$ J
  - Touch at 1cm distance: $U_{grav} = -3.34 \cdot 10^{-7}$ J
  - Total kinetic energy increases by $3.0 \cdot 10^{-7}$ J
  - Mass m will have twice the speed as mass M (momentum conservation)
    => K.E. = $m/2\, v^2 + M/2\, v^2/4 = 3/4\, mv^2$
    => v = $8.9 \cdot 10^{-5}$ m/s (< 0.1 mm/s) (**not** a constant acceleration!)
- Escape velocity from Earth:
  - First $U_{grav} = -GmM_{Earth}/R_{Earth}$, K.E. = $m/2\, v^2$
  - Finally, $U_{grav} = 0$ and K.E. = 0
  - Energy conserved
    => $GmM_{Earth}/R_{Earth} = m/2\, v^2$
    => $v^2 = 2GM_{Earth}/R_{Earth} = 2gR_{Earth}$ => v = 11,200 m/s

Extreme Example: Black holes

- Make radius R smaller and smaller while keeping M constant:
  - Collapsing stars (supernovae)
  - Large initial density fluctuation in the Universe ("primordial black holes", quasars); center of most (all?) galaxies
  - High energy collisions?
- $v^2 = 2GM/R$ becomes larger and larger, until it exceeds the speed of light c^2 => NOTHING can escape once it is closer than R to the center
- All matter crushed to "infinitely small" center inside black hole
- Huge tidal forces because of $1/R^3$
- Complete description requires Einstein's GENERAL theory of relativity (curvature of space-time)