Astrophysics Lecture Notes

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Pop Quiz!!

Is the universe homogeneous?
    Roughly! If averaged over scales of 10^9 *pc.*
Is the universe isotropic?
    Roughly! If you average over large enough angular range and ignore the foreground.
Is the universe static?
    No! The universe is constantly changing in many ways.

Lecture Notes

We know that the universe is expanding. We can express that with the so called Hubble Constant, which refers to the time of today. This is given by the notation H_0. More importantly, the Hubble Constant is not actually a constant. It changes with time, which we will call the Hubble Parameter.

$$H(t) = \text{Hubble Parameter}$$
$$H(t_0) = H_0 \text{ where } t_0 \text{ represents today.}$$

Even if the velocity were exactly constant, since you have to divide by a length scale all length scales were shorter in the early universe, the H was bigger in earlier times of the universe. This leads to one of many violations of special relativity.

Violations of Special Relativity

- There IS a perferred inertial system at any given point! It it the point where the CMB does not have a dipole modulation. That is, it is at rest with respect to the CMB.
- There IS a universal time! Anywhere in the universe there is a way to define an absolute time. This is defined by the Hubble Constant(Parameter), H(t). The hubble constant itself is measured in units of $\frac{1}{time}$. With this, anything else can be defined as a function of time.
- Speeds greater than the speed of light!

Brief History of the Universe

At some point in time ($t > 0$) there was an event, which we call The Big Bang. Any time before that or even at time t = 0 is meaningless, in that we don't know how to represent or calculate anything about it in any meaningful way. The earliest time that we can make some predictions about is $t = 10^{-37} seconds$ after The Big Bang, when inflation began. Inflation is the sudden and enormous (a factor of 10^{28}) expansion of the universe in about 10^{-35} seconds to approximately the size of a golf ball. Initially it was filled with elementary particles that constantly collided: a primordial soup of elementary particles. They would eventually combine to form protons and neutrons, which would then combine to form nuclei. Keep in mind that this takes place all in the first few minutes and it set the ground for how many protons, helium atoms, etc. were around in the beginning. After 400,000 years the "soup" cooled down enough for these nuclei to combine with electrons and the universe became filled with a gas of neutral atoms and became transparent to light. The CMB is from photons emitted at this moment.

After a few hundred million years (maybe less) the universe started to clump and form dark matter, which formed structures that attracted gas and dust and eventually formed stars. This is called the Era of Reionization, due to the massive stars producing enough ultra-violet light to re-ionize some of the gas. Some time after (a billion years or so), the first galaxies formed and the universe looked almost the same as it does today.

Defining Fixed Positions in a Universal Coordinate System

How do we describe the universe as a whole in a way that is consistent with General Relativity? GR acts by modifying the distance between two points by use of a metric. We need to find the metric of the universe *as a whole*.

As there is a preferred inertial system at any given point, we must practice caution. Whatever we call our preferred inertial system is not the same for a galaxy moving away from us at $0.1c$ or even $0.01c$. That galaxy will have its own preferred inertial system which is at rest with respect to what it sees from the CMB. If we think about any point in the universe, the tricky part is defining that point's position. The idea is to give it a position that does not change over time *if* that point is in its own preferred inertial system. This coordinate system, labeled a co-moving coordinate system, is defined by the dimensionless quantity $\vec{r_c}$. Instead of saying that some galaxy is however far away in the direction of the constellation Taurus, which as we know would not remain constant for many reasons, it would instead be given an absolute coordinate $\vec{r_c}$ that does not change over time. We do this by taking the distance to this object and divide it by a time dependent scale factor $a(t)$.

$$\vec{r_c} = \frac{\vec{d}}{a(t)}$$

Instead of keeping track of how every object moves with the expansion of the universe, we instead give an object a fixed position from which we can determine its distance from us.

$$\vec{r_c} \cdot a(t) = \vec{d} = c \cdot \Delta t$$

The Hubble Parameter

What is the value of the scale factor? The best way to go about it is that it is completely arbitrary. But, if we assume the universe has a radius (it's curved), then we just choose $a(t)$ to be the radius. If the universe has no radius (it's flat), then $a(t)$ is arbitrary. There are only three possibilities for the shape of the universe (which will be expanded on in a later lecture):

- Flat, infinite Universe of no curvature. $K = 0$.
- Spherical, finite (closed) Universe of positive curvature. $K = +1$. This is the only case where $\dot{a}(t) = 0$ is possible.
- Infinite (open) Universe of negative curvature (”Saddle” Universe). $K = -1$. This is the only case where $\dot{a}(t) = constant$ is at least possible.

If the distance an object has from us at some time is given by

$$\vec{d}(t) = a(t) \cdot \vec{r_c}$$

then how fast it is moving away can be found by

$$\frac{d}{dt}\vec{d}(t) = \frac{d}{dt}[a(t) \cdot \vec{r_c}] = \dot{a}(t) \cdot \vec{r_c}$$

$$\dot{\vec{d}}(t) = \dot{a}(t) \cdot \vec{r_c} = \dot{a}(t) \cdot \frac{\vec{d}(t)}{a(t)} = \vec{d}(t) \cdot \frac{\dot{a}(t)}{a(t)} = H(t)\vec{d}(t)$$

The velocity with which something moves away from us must be proportional to how far away from us it is. The quantity

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

is the **Hubble Parameter**, the rate of expansion of the universe. It is easily seen that it is going to change with time, unless a(t) is a constant. There is a connection between a(t) and the content of the universe, in fact, a(t) is completely determined by what's inside the universe. Einstein came up with a cosmological constant to make a(t) constant and later, after Mr. Hubble discovered that the universe was in fact expanding, he scrapped the idea and instead set to find a solution to the laws of General Relativity for an expanding universe. Today, it turns out that we DO need a cosmological constant; however, we refer to it as Dark Energy (more on Dark Energy and its effects on the universe next lecture).

Hubble's law, being an exact solution, lends way to a violation of special relativity: speeds greater than the speed of light. No matter how big H_0 may be, there is some distance away that the rate of expansion will exceed the speed of light. How do we reconcile this expanding scale factor with the observed redshift?

Relating the Scale Factor and Relativistic Redshift

The relativistic equation for redshift

$$z = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - 1 = \frac{\lambda_{obs}}{\lambda_{emit}} - 1 \approx \frac{v}{c}$$

has a limit for the velocity such that $v = c$. For any possible, real value of z, the velocity has to be less than the speed of light. To solve this problem, we relate it instead to the scale factor of the universe. To visualize this relationship, consider the following thought experiment:

> Picture a moving walkway that is also being stretched in one direction. Along the walkway there is a nicely inscribed distance scale (which corresponds to $\vec{r_c}$). Imagine a bunch of people hopping onto the walkway at fixed intervals (once every period of an optical oscillation, *i.e.* with a frequency ν.) If they all hop on during a short piece of time, they all have the same separation $\Delta\vec{r_c}$, since the band hasn't stretched appreciably. These people are supposed to represent the peaks and valleys of a light wave. Each peak and each valley will move with the speed of light, which means that their distance in co-moving coordinates stays constant:

$$\Delta d = c \cdot \Delta t$$

$$\Delta d = \Delta\vec{r_c} \cdot a(t)$$

$$\frac{\Delta\vec{r_c}}{\Delta t} = \frac{c}{a(t)}$$

> That is, the *true* distance between the peaks and valleys increases with time, proportional to $a(t)$.As the universe expands, light becomes slower and slower as measured in co-moving coordinates because the distance between any two co-moving coordinates is increasing. For the few wiggles of a lightwave over a few hundred nanometers within a tiny fraction of a second, the $\frac{\Delta(\vec{r_c})}{\Delta t}$ is the same for all valleys and peaks of that wave.

Ultimately, in terms of the co-moving coordinates the light wave has fixed wavelength. That means

$$\frac{\lambda}{a(t)} \approx constant$$

Simply put, once light reaches our position we get a wavelength that has been stretched by the same amount as the increase in size of the universe. We can write

$$\frac{\lambda_{obs}}{\lambda_{emit}} = \frac{a(t_{obs})}{a(t_{emit})} = z + 1$$

It turns out that this means the redshift is also proportional to the hubble constant. To find a time when light was emitted, we start with

$$\frac{a(t_{obs})}{a(t_{emit})} = z + 1$$

Through a little algebra we obtain:

$$a(t_{obs}) - a(t_{emit}) = z \cdot a(t_{emit})$$

The left side is equal to the integral over the time derivative:

$$a(t_{obs}) - a(t_{emit}) = \int_{t_{emit}}^{t_{obs}} \dot{a}(t)dt$$

$$\int_{t_{emit}}^{t_{obs}} \dot{a}(t)dt = \frac{z}{z+1} a(t_{obs})$$

Which can be solved for t_{emit} if $\dot{a}(t)$ is known. The coefficient of $a(t_{obs})$ is the fractional increase of the universe since emission, compared to today. We can use the equation for the speed of light to calculate where that object was when the light was emitted and where the object is today. To find the position when light was emitted:

$$\frac{d\vec{r_c}}{dt} = \frac{c}{a(t)}$$

$$\Delta\vec{r}_{c_{emit}} = \int_{t_{emit}}^{t_0} \frac{c \cdot dt}{a(t)}$$

We can multiply the result with at_{emit} to find the actual distance the object had from us at the time of emission. Since the position in co-moving coordinates does not change with time, we can use that to find out how far the object is away from us today, by simply multiplying by the size of the universe today:

$$\Delta\vec{r}_{c_{emit}} \cdot a(t_0) = \Delta d.$$

If we assume that the size of the universe increases by the same amount every year (meaning $\dot{a}(t)$ is constant) then the value of today's hubble constant tells us exactly when the universe began. (This scenario is roughly correct for a Universe that is mostly dominated by negative curvature, with little "stuff" in it.) Keep in mind that even though $\dot{a}(t)$ is constant it does not mean that the Hubble parameter is time independent, because $a(t)$ still changes with time. Remember:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Which is a velocity over a distance, and as we know velocity is merely a distance over a change in time. This simplifies to be a unit of $\frac{1}{t}$. Simply put, the inverse of H_0 is the age of the universe in this case, $t_0 = 1/H_0$, and we can write:

$$a(t) = \dot{a} \cdot t = a_0 H_0 t.$$

In general, for a constant $\dot{a}(t)$ (which implies negative curvature), the integral

$$\int_{t_e}^{t_0} \dot{a}(t)dt$$

is just the time difference multiplied by $\dot{a}$. From the equation above:

$$\int_{t_{emit}}^{t_{obs}} \dot{a}(t)dt = \frac{z}{z+1} a(t_0)$$

$$\dot{a}(t_0 - t_e) = \frac{z}{z+1} a(t_0)$$

$$t_0 - t_e = \frac{z}{z+1}\frac{a(t_0)}{\dot{a}} = \frac{z}{z+1}\frac{1}{H_0}$$

To find the distance to a light emitter yields:

$$\Delta\vec{r}_{c_{emit}} = \int_{t_{emit}}^{t_0} \frac{c \cdot dt}{a_0 H_0 t} = \frac{c}{a_0 H_0} \ln \frac{t_0}{t_{emit}}.$$

Since

$$\frac{t_0}{t_{emit}} = \frac{\dot{a}t_0}{\dot{a}t_{emit}} = \frac{a_0}{a(t_{emit})} = z+1$$

we can conclude that the distance today is

$$D(em., t_0) = a_0 \Delta\vec{r}_{c_{emit}} = \frac{c}{H_0} ln(z+1)$$

and the distance at the time of emission

$$D(em., t_{emit}) = \frac{c}{H_0(z+1)} ln(z+1)$$

Walker-Robinson Metric

This metric, given by

$$ds^2 = dt^2 - a^2(t)[dr_c^2 + S_K^2(r_c)(d\theta^2 + sin^2\theta d\phi^2)]$$

is the distance between two points while accounting for the curvature for the universe.

- Closed Universe, positive curvature ($K = +1$)
$$S_K(r_c) = sin(r_c)$$
- Flat Universe, no curvature ($K = 0$)
$$S_K(r_c) = r_c$$
- Open Universe, negative curvature ($K = -1$)
$$S_K(r_c) = sinh(r_c)$$