Derivation of the Virial Theorem

Theorem:

Averaged over long enough times, the average kinetic energy of a gravitationally bound system of particles is ½ in magnitude (and of course opposite sign) of the gravitational potential energy of the system.

Proof:

Begin by defining a quantity

$$I = \sum_i m_i \vec{r}_i^{\,2}$$

where the sum goes over all particles of the system. This quantity could be interpreted as the weighted mean square distance of the system from the origin (which we can choose to be the center of mass of the system), multiplied by the total mass of the system:

$$I / M = \langle \vec{r}^{\,2} \rangle$$

However, the interpretation is not as important as the manipulations we can do with it:

$$\frac{dI}{dt} = 2\sum_i m_i \vec{r}_i \cdot \dot{\vec{r}}_i \Rightarrow \quad \frac{1}{2}\frac{d^2 I}{dt^2} = \sum_i m_i \dot{\vec{r}}_i^{\,2} + \sum_i m_i \vec{r}_i \cdot \ddot{\vec{r}}_i = 2T_{kin} + \sum_i \vec{r}_i \cdot \vec{F}_i$$

The second term can be expanded, using the gravitational law of attraction between all possible pairs of particles (*i,j*) in the system:

$$\vec{F}_i = \sum_{j\neq i} G \frac{m_i m_j}{\left|\vec{r}_j - \vec{r}_i\right|^3}\left(\vec{r}_j - \vec{r}_i\right) \Rightarrow \quad \frac{1}{2}\frac{d^2 I}{dt^2} = 2T_{kin} + \sum_{i,j;i\neq j} G \frac{m_i m_j}{\left|\vec{r}_j - \vec{r}_i\right|^3}\left(\vec{r}_j - \vec{r}_i\right)\cdot \vec{r}_i = 2T_{kin} +$$

$$\left\{ \sum_{i,j;i\neq j} G \frac{m_i m_j}{\left|\vec{r}_j - \vec{r}_i\right|^3}\left(\vec{r}_j - \vec{r}_i\right)\cdot \left[\frac{\left(\vec{r}_i - \vec{r}_j\right)}{2} + \frac{\left(\vec{r}_i + \vec{r}_j\right)}{2}\right] = \sum_{i,j;i\neq j} -\frac{G}{2}\frac{m_i m_j}{\left|\vec{r}_j - \vec{r}_i\right|^3}\left(\vec{r}_j - \vec{r}_i\right)^2 + \sum_{i,j;i\neq j} \frac{G}{2}\frac{m_i m_j}{\left|\vec{r}_j - \vec{r}_i\right|^3}\left(\vec{r}_j^{\,2} - \vec{r}_i^{\,2}\right) \right\}$$

The first sum on the r.h.s. of the bottom line is simply the total gravitational potential energy of the system: The square cancels with the cube to give just the distance between the two particles (*i,j*) and since the sum goes over all pairs twice (once with $i<j$ and once with $i>j$), one has to divide by 2 to get the total potential energy. The second sum adds up to zero, since for each pair (*i,j*) that enters, the reversed pair (*j,i*) also enters but with opposite sign. Integrating both sides over a (long) time interval T, and dividing the result by T, we find

$$\frac{1}{T}\int_0^T \frac{1}{2}\frac{d^2 I}{dt^2}dt = \frac{1}{T}\int_0^T 2T_{kin}\,dt + \frac{1}{T}\int_0^T V_{pot}\,dt = \langle 2T_{kin}\rangle + \langle V_{pot}\rangle$$

where the last two terms are the averages of kinetic and potential energy over the time T. The integral on the l.h.s. can be integrated to yield

$$\frac{1}{T}\int_0^T \frac{1}{2}\frac{d^2 I}{dt^2}dt = \frac{1}{2T}\left(\frac{dI}{dt}(T) - \frac{dI}{dt}(0)\right) = \frac{1}{T}\sum_i m_i \left[\vec{r}_i(T)\cdot\dot{\vec{r}}_i(T) - \vec{r}_i(0)\cdot\dot{\vec{r}}_i(0)\right]$$

If, as we assumed, the system is gravitationally **bound**, the expression inside the sum cannot grow "without bounds" as T increases – it must remain finite. Dividing by T will therefore yield a quantity that goes to zero at sufficiently large times. Therefore, the sum of twice the kinetic energy plus the potential energy of the system averages to zero over such large times, i.e. $\langle T_{kin}\rangle = -\frac{1}{2}\langle V_{pot}\rangle$, q.e.d.