Classical Mechanics.

Lecture Notes

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Part I

1 Basic Principles

1.1 Newton Laws

Inertial reference frame is a frame where every body is at rest or in uniform motion unless acted on by external force.

Practical definition: Inertial reference frame = reference frame at rest or in uniform motion with respect to distant stars

Newton's first law actually states that the inertial reference frames do exist. Actually, there is an infinite number of them.

Newton's second law: In an inertial frame rate of change of momentum of a certain body is equal to (vector) sum of all forces acting on it.

$$\frac{d\vec{p}}{dt} = \sum_i \vec{F}_i \tag{1.1}$$

Here $\vec{p} = m\vec{v}$ is a momentum and $\vec{F}_i$ are forces acting on the body.

If mass m is constant,

$$m\frac{d\vec{v}}{dt} \equiv m\vec{a} = \sum_i \vec{F}_i \tag{1.2}$$

Note that Newton's 2nd law is commonly cited as $\sum \vec{F} = m\vec{a}$ but the most general form is actually Eq. (1.1). It can be used to study motion of system with variable mass such as a rocket or an evaporating droplet.

Newton's third law: To each action, there is a reaction of equal magnitude but opposite direction

$$\vec{F}_{12} = -\vec{F}_{12} \tag{1.3}$$

where F_{ij} is a force exerted on body "j" by body "i". Note that the forces in action-reaction pair act on different bodies.

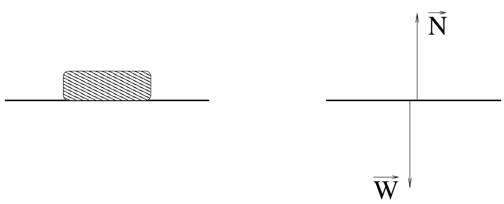

Figure 1. Action-reaction pair for the brick on the surface.

Here $\vec{W}$ is a force of weight exerted by the brick on the surface and $\vec{N}$ normal force exerted on the brick by the surface.

For the interaction of two particles the forces in the action-reaction pair act along the line separating particles. As an example, we may consider gravitational

$$\vec{F}_{12} \;=\; Gm_1m_2\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|^3}, \qquad \vec{F}_{21} \;=\; Gm_1m_2\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_1-\vec{r}_2|^3}, \qquad \vec{F}_{12}=-\vec{F}_{12} \tag{1.4}$$

or Coulomb forces

$$\vec{F}_{12} \;=\; -\frac{q_1q_2}{4\pi\epsilon_0}\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|^3}, \qquad \vec{F}_{21} \;=\; -\frac{q_1q_2}{4\pi\epsilon_0}\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_1-\vec{r}_2|^3}, \qquad \vec{F}_{12}=-\vec{F}_{12} \tag{1.5}$$

1.2 Conservation Laws

Linear momentum

$$\sum_n \vec{F}_n \;=\; 0 \;\;\Rightarrow\;\; \vec{p} \;=\; \text{const} \tag{1.6}$$

Note that the vector nature of this equation, in particular p_i is conserved if $\sum(\vec{F}_n)_i \;=\; 0$. Example: projectile motion (see Fig. 2).

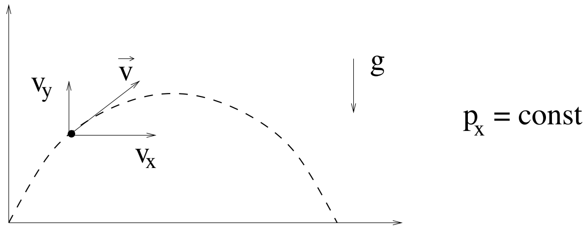

Figure 2. Projectile motion.

Angular momentum

Angular momentum of a particle (with respect to origin) is defined as

$$\vec{L} \;=\; \vec{r}\times\vec{p} \;=\; m\vec{r}\times\vec{v} \tag{1.7}$$

where $\vec{r}$ is the position of the particle and $\vec{p}$ is the momentum.

If mass of the particle is constant

$$\frac{d\vec{L}}{dt} \;=\; m\dot{\vec{r}}\times\vec{v} \;+\; m\vec{r}\times\dot{\vec{v}} \;=\; m\vec{v}\times\vec{v} \;+\; m\vec{r}\times\vec{a} \;=\; \vec{r}\times\sum_n\vec{F}_n \tag{1.8}$$

The quantity in the r.h.s. is called torque

$$\vec{\tau} \;=\; \sum_n \vec{\tau}_n, \qquad \vec{\tau}_n \;\equiv\; \vec{r}\times\vec{F}_n \tag{1.9}$$

so

$$\dot{\vec{L}} \;=\; \sum_n \vec{\tau}_n \tag{1.10}$$

and if $\sum_n \vec{\tau}_n \;=\; 0$ the angular momentum is conserved.

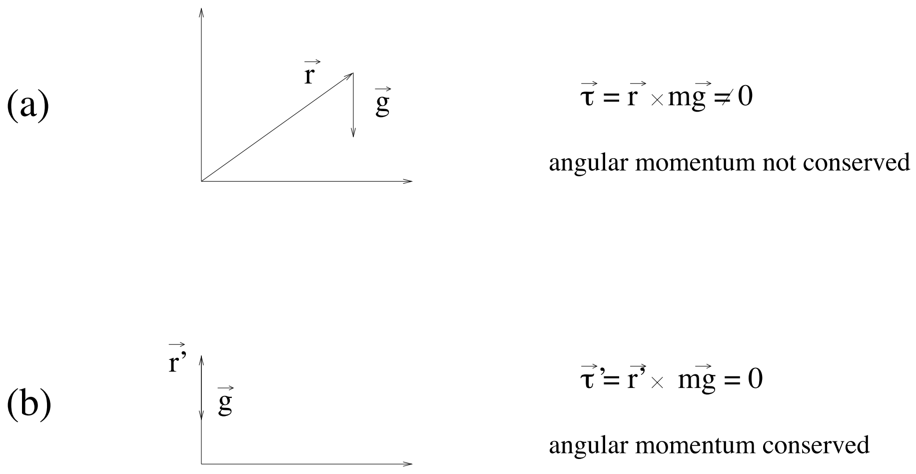

Figure 3. Angular momentum of a particle in a free fall.

Let me note again that both angular momentum and torque depend on the choice of the origin of coordinate frame.

Example: free fall

Figure a: $\vec{\tau} \;=\; m\vec{r}\times\vec{g} \;\neq\; 0 \Rightarrow \vec{L}$ is not conserved.
Figure b: $\vec{\tau}' \;=\; m\vec{r}'\times\vec{g} \;=\; 0 \Rightarrow \vec{L}'$ is conserved.

<u>Work and energy</u>

Suppose a particle moves from point A to point B along some path.

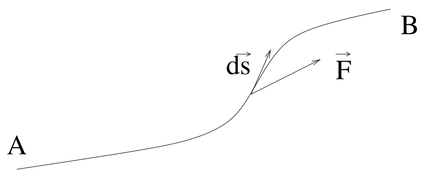

Figure 4.

The work done by a force $\vec{F}$ acting on the particle is given by

$$W^{A\to B} \;=\; \int_B^A d\vec{s}\cdot\vec{F} \tag{1.11}$$

where $d\vec{s} = \vec{v}(t)dt$ is an infinitesimal displacement.

Note that if $\vec{F} \;=\; \sum_n \vec{F}_n$ the total work is a sum of the works done by individual forces

$$W^{A\to B} \;=\; \int_A^B d\vec{s}\cdot\sum_n\vec{F}_n \;=\; \sum_n\int_A^B d\vec{s}\cdot\vec{F}_n \;=\; \sum W_n^{A\to B} \tag{1.12}$$

The work done by the force leads to the change in kinetic energy of the particle

$$W^{A\to B} = \int_A^B d\vec{s}\cdot\vec{F} = m\int_A^B \frac{d\vec{s}}{dt}dt\cdot\frac{d\vec{s}}{dt^2} = \frac{m}{2}\int_{t_A}^{t_B} dt\frac{d}{dt}(\frac{d\vec{s}}{dt})^2 = \frac{mv_B^2}{2}-\frac{mv_A^2}{2} = T_B - T_A \tag{1.13}$$

Conservative forces

If $\oint_C \vec{s}\cdot\vec{F}$ vanishes for any closed contour C then $\vec{F}$ is said to be a conservative force The well-known examples are gravitational and Coulomb forces:

$$\oint_C d\vec{r}\cdot\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} = \lim_{\Delta r_i\to 0}\sum_{i=1}^N \Delta\vec{r}_i\cdot\frac{\vec{r}_i-\vec{r}_0}{|\vec{r}_i-\vec{r}_0|^3} = -\lim_{\Delta r_i\to 0}\sum_{i=1}^N \Delta\frac{1}{|\vec{r}_i-\vec{r}_0|} = 0 \tag{1.14}$$

Similarly, the well-known counterexample is a force of friction: the work done by the force of friction along any path i(closed or not) s always positive.

Stokes theorem:

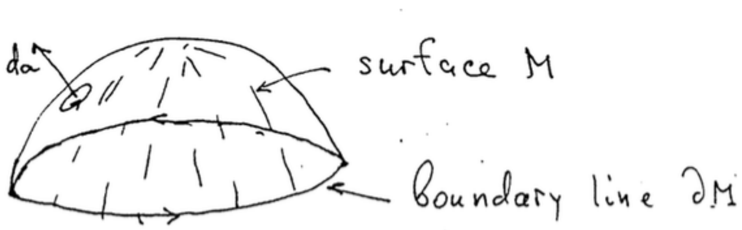

Figure 5. Stokes theorem.

$$\oint_M d\vec{a}\cdot\vec{\nabla}\times\vec{v} = \oint_{C=\partial M} d\vec{s}\cdot\vec{v} \tag{1.15}$$

Consequently, $\oint_C d\vec{s}\cdot\vec{F} = 0$ for any loop C means that $\vec{\nabla}\times\vec{F} = 0$ which implies that $\vec{F}$ is a gradient of some scalar function U called the potential energy

$$\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r}) \tag{1.16}$$

where minus stands for historical reasons. (Recall that $\vec{\nabla}\times\vec{\nabla}f(\vec{r}) = 0$ for any scalar function $f(\vec{r})$).

The work done by a conservative force

$$W_{A\to B} = \int_A^B d\vec{s}\cdot\vec{F}(\vec{r}) = \int_A^B dU(\vec{r}) = -U(\vec{r}_B)+U(\vec{r}_A) = -U_B+U_A \tag{1.17}$$

is independent of path taken between points A and B.

As we saw above (Eq. (1.13)) the work done by a sum of forces (conservative and/or non-conservative) is equal to a change in kinetic energy so

$$\begin{aligned}\sum_n W_{A\to B,n} &= \sum_{\rm conserv} W_{A\to B,n} + \sum_{\rm non-conserv} W_{A\to B,n} = T_B - T_A \\ \Leftrightarrow \sum_n W_n^{A\to B} &= \sum_{\rm conserv} (U_n^A - U_n^B) + \sum_{\rm non-conserv} W_n^{A\to B} = T_B - T_A \end{aligned} \tag{1.18}$$

and therefore

$$T^A + U^A + \sum_{\rm non-conserv} W_n^{A\to B} = T_B + U^B \tag{1.19}$$

where $U^A \equiv \sum_{\rm conserv} U_n^A$ (and $U^B \equiv \sum_{\rm conserv} U_n^B$). The quantity $T + U$ (the sum of kinetic and potential energies of the particle) is called the total energy of the particle. If non-conservative forces do no work, the total energy is conserved

$$T^A + U^A = T^B + U^B \tag{1.20}$$

1.3 Systems of particles

Consider N particles with masses m_n [1] and positions $\vec{r}_n$ in an inertial reference frame. Define the position $\vec{R}$ of center of mass by

$$\vec{R} = \frac{1}{M}\sum_n m_n\vec{r}_n, \qquad M = \sum_n m_n \tag{1.21}$$

The total momentum of this system of particles is a sum of the individual momenta of each particle

$$\vec{P} = \sum_n \vec{p}_n = \sum_n m_n\vec{v}_n \tag{1.22}$$

1.3.1 Center of mass motion

For each particle, the 2nd law of Newton reads

$$\dot{\vec{p}}_n(t) = \sum \vec{F}_n \tag{1.23}$$

Among forces $\vec{F}_n$ acting on particle "n" there are external forces $\vec{F}_n^{\rm ext}$ due to agents outside the system and internal forces $\vec{F}_n^{\rm int} = \sum_{m\neq n} \vec{F}_{mn}$ ewhere $\vec{F}_{mn}$ is a force which particle "m" exert on particle "n".

$$\dot{p}_n(t) = \vec{F}_n^{\rm ext} + \sum_{m\neq n} \vec{F}_{mn} = \vec{F}_n^{\rm ext} + \frac{1}{2}\sum_{m\neq n}(\vec{F}_m n + \vec{F}_{nm}) = \sum \vec{F}_n^{\rm ext} \tag{1.24}$$

Now, differentiating Eq. (1.22) with respect to t we obtain

$$\begin{aligned}\dot{\vec{P}} &= \sum_n m_n\dot{\vec{p}}_n(t) = \sum_n \vec{F}_n = \sum_n \vec{F}_n^{\rm ext} + \sum_n\sum_{m\neq n}\vec{F}_{mn} \\ &= \sum_n \vec{F}_n^{\rm ext} + \frac{1}{2}\sum_{m\neq n}(\vec{F}_{mn} + \vec{F}_{nm}) = \sum_n \vec{F}_n^{\rm ext}\end{aligned} \tag{1.25}$$

[1] From now on, unless specified, the mass of each particle is assumed to be constant

where we used Newton's third law $\vec{F}_{mn} = -\vec{F}_{nm}$. Thus, the center of mass motion is not affected by the internal forces and is determined by action of all external forces.

Note that Eq. (1.25) is a vector equation corrrect for each component separately so if the projection of external forces on some axis vanishes the corresponding component of c.m. momentum $\vec{P}$ is conserved

Example:

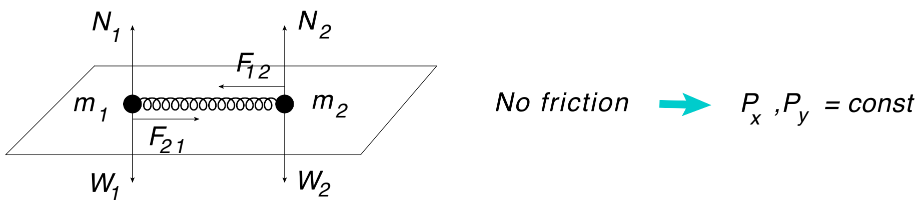

Figure 6. Two masses connected by a spring on x, y plane.

1.3.2 Angular momentum

Total angular momentum for a system of particles is defined as sum of the individual angular momenta

$$\vec{L} \equiv \sum \vec{L}_n = \sum \vec{r}_n \times \vec{p}_n \tag{1.26}$$

Differentiating this equation with respect to time one obtains

$$\dot{\vec{L}} = \sum_n m_n \dot{\vec{L}}_n(t) = \sum_n (\vec{r}_n \times \dot{\vec{p}}_n + \dot{\vec{r}}_n \times \vec{p}_n)$$
$$= \sum_n \vec{r}_n \times \left(\vec{F}_n^{\rm ext} + \sum_m \vec{F}_{mn}\right) = \sum_n \vec{r}_n \times \vec{F}_n^{\rm ext} + \sum_n \sum_{m\neq n} \vec{r}_n \times \vec{F}_{mn} = \sum_n \vec{\tau}_n^{\rm ext} + \sum_{m\neq n} \vec{r}_n \times \vec{F}_{mn}$$

If the forces in action-reaction pair are along the line connecting two particles, we get

$$\dot{\vec{L}} = \sum_n \vec{\tau}_n^{\rm ext} \tag{1.27}$$

because

$$\sum_n \sum_{m\neq n} \vec{r}_n \times \vec{F}_{mn} = \frac{1}{2} \sum_{m\neq n} (\vec{r}_n \times \vec{F}_{mn} + \vec{r}_m \times \vec{F}_{nm}) = \frac{1}{2} \sum_{m\neq n} (\vec{r}_n - \vec{r}_m) \times \vec{F}_{mn} = 0 \tag{1.28}$$

NB: the assumption that forces are aligned with the separation between particles is not universal - for example, it is not true for general electromagnetic forces of moving particles.

Part II

1.3.3 Decomposition of $\vec{L}$ into $\vec{L}_{\rm c.m.}$ and $\vec{L}'$

Let us decompose vector $\vec{r}_n$ for each particle into a sum of vector $\vec{R}$ and separation from the c.m. $\vec{r}'$:

$$\vec{r}_n = \vec{R} + \vec{r}'_n, \quad \Rightarrow \quad \vec{v}_n = \dot{\vec{r}}_n = \dot{\vec{R}} + \dot{\vec{r}}'_n \equiv \vec{V} + \vec{v}'_n \tag{1.29}$$

where $\vec{R}$ is a position of center of mass and $\vec{V} = \dot{\vec{R}}$ is its velocity.

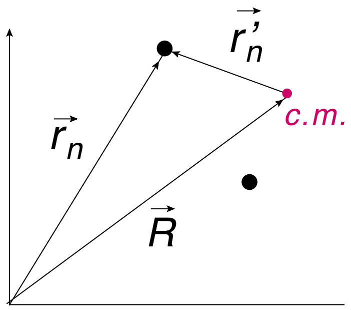

Figure 7.

Note that

$$\sum_n m_n \vec{r}'_n \;=\; \sum_n m_n(\vec{r}_n - \vec{R}) \;=\; \sum_n m_n \vec{r}_n - M\vec{R} \;=\; 0 \qquad (1.30)$$

due to the definition of the center of mass (1.21). This also implies that

$$\sum_n m_n \vec{v}'_n \;=\; \sum_n m_n \dot{\vec{r}}'_n \;=\; \frac{d}{dt}\sum_n m_n \vec{r}'_n \;=\; 0 \qquad (1.31)$$

Using these formulas one can rewrite $\vec{L}$ with respect to an arbitrary origin in terms of $\vec{L}'$ w.r.t. center of mass:

$$\vec{L} \;=\; \sum m_n \vec{r}_n \times \vec{v}_n \;=\; \sum_n m_n(\vec{R} + \vec{r}'_n) \times (\vec{V} + \vec{v}'_n) \qquad (1.32)$$

$$=\; M\vec{R}\times\vec{V} + \vec{R}\times\sum m_n \vec{v}'_n + \sum_n m_n \vec{r}'_n \times \vec{V} + \sum m_n \vec{r}'_n \times \vec{v}'_n \;=\; M\vec{R}\times\vec{V} + \vec{L}'$$

Now, from Eq. (1.27) we get

$$\dot{\vec{L}}_{\text{c.m.}} + \dot{\vec{L}}' \;=\; \sum_n \vec{\tau}_n^{\,\text{ext}} \qquad (1.33)$$

On the other hand

$$\dot{\vec{L}}_{\text{c.m.}} \;=\; \dot{\vec{R}}\times\vec{P} + \vec{R}\times\dot{\vec{P}} \;=\; \vec{R}\times\sum_n \vec{F}_n^{\text{ext}} \qquad (1.34)$$

and therefore

$$\dot{\vec{L}}' \;=\; \sum_n \vec{\tau}_n^{\,\text{ext}} - \dot{\vec{L}}_{\text{c.m.}} \;=\; \sum_n \vec{\tau}_n^{\,\text{ext}} - \vec{R}\times\dot{\vec{P}} \;=\; \sum_n (\vec{r}_n - \vec{R})\times\vec{F}_n^{\text{ext}} \qquad (1.35)$$

so

$$\dot{\vec{L}}' \;=\; \sum_n \vec{r}'_n \times \vec{F}_n^{\text{ext}} \qquad (1.36)$$

The rate of change of angular momentum $\vec{L}'$ is equal to the sum of external torques about center of mass. **NB**: this relation holds for arbitrary motion of center of mass, even in the case if it is accelerating (a frame attached to c.m. would not be inertial in this case).

1.3.4 Work and energy

$$W^{A\to B} \;=\; \sum_n \int_B^A d\vec{s}\cdot\vec{F}_n \tag{1.37}$$

where $d\vec{s} = \vec{v}(t)dt$ is an infinitesimal displacement.

Note that if $\vec{F} \;=\; \sum_n \vec{F}_n$ the total work is a sum of the works done by individual forces

$$W^{A\to B} \;=\; \sum_n \int_A^B d\vec{s}_n\cdot\vec{F}_n \;=\; \sum_n \int_B^A d\vec{s}\cdot\vec{F}_n \;=\; \sum W_n^{A\to B} \tag{1.38}$$

Now, $d\vec{s}_n \;=\; \vec{v}_n dt$ and $m_n\dot{v}_n \;=\; \vec{F}_n$ so

$$W^{A\to B} \;=\; \sum_n \int_A^B dt\;\vec{v}_n\cdot\big(m_n\dot{\vec{v}}_n\big) \;=\; \sum_n \frac{m_n}{2}\int_A^B dt\;\frac{d\vec{v}_n^2}{dt} \;=\; \sum_n \big(T_n^B - T_n^A\big) \;=\; T_B - T_A \tag{1.39}$$

where $T = \sum_n \frac{m_n}{2} v_n^2$ is the sum of the kinetic energies of particles. Note that

$$T \;=\; \sum_n \frac{m_n}{2} v_n^2 \;=\; \sum_n \frac{m_n}{2}(v_n' + V)^2 \;=\; \sum \frac{m_i}{2} v_i^2 + \frac{M}{2}V^2 \;=\; T' + T_{\rm c.m.} \tag{1.40}$$

where we used Eq. (1.31). Next, assume that both external and internal forces are conservative, then

$$\vec{F}_i^{\rm ext}(\vec{r}_n) \;=\; -\vec{\nabla}_i V^{\rm ext}(\vec{r}_n) \tag{1.41}$$

and

$$\vec{F}_{ji}(\vec{r}_n) \;=\; -\vec{\nabla}_i V^{\rm ext}(\vec{r}_{ij}) \;=\; -\frac{\vec{\partial}}{\partial r_{ij}} V(\vec{r}_{ij}) \tag{1.42}$$

Note that in order to satisfy Newton's 3rd law $V(\vec{r}_{ij})$ must be a function of the magnitude r_{ij}: $V(\vec{r}_{ij}) = V(|\vec{r}_{ij}|)$. Indeed, since $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$

$$\vec{F}_{ij} \;=\; -\vec{\nabla}_j V^{\rm ext}(r_{ij}) \;=\; \frac{\vec{\partial}}{\partial r_{ij}} V(r_{ij}) \;=\; -\vec{F}_{ji} \tag{1.43}$$

The formula for the work (1.38) takes the form ($\vec{\nabla}_{ij} \equiv \frac{\vec{\partial}}{\partial r_{ij}}$):

$$\begin{aligned} W^{A\to B} \;&=\; -\sum_n \int_B^A d\vec{r}_n\cdot\nabla_n V^{\rm ext}(\vec{r}_n) - \sum_{i,j}\int_B^A d\vec{r}_i \frac{\vec{\partial}}{\partial r_{ij}} V(r_{ij}) \\ &=\; -\sum_n V^{\rm ext}(\vec{r}_n)\Big|_A^B \;-\; \frac{1}{2}\sum_{i,j}\int_A^B \big(d\vec{r}_i\cdot\vec{\nabla}_{ij}V_{ij} + d\vec{r}_j\cdot\vec{\nabla}_{ji}V_{ij}\big) \end{aligned} \tag{1.44}$$

However, due to $V(r_{ij}) = V(r_{ji})$ and $\vec{\nabla}_{ij} = -\vec{\nabla}_{ji}$ the second term in the above equation can be rewritten as

$$-\;\frac{1}{2}\sum_{i,j}\int_A^B (d\vec{r}_i - d\vec{r}_j)\cdot\vec{\nabla}_{ij}V(r_{ij}) \;=\; -\frac{1}{2}\sum_{i,j} V(r_{ij})\Big|_A^B \tag{1.45}$$

and therefore

$$W^{A\to B} \;=\; \sum_n \big(V_A^{\rm ext}(\vec r_n) - V_B^{\rm ext}(\vec r_n)\big) \;+\; \frac{1}{2}\sum_{i,j}\big(V_A(r_{ij}) - V_B(r_{ij})\big) \tag{1.46}$$

Since $W^{A\to B} = T^B - T^A$ (see Eq. (1.39)) we get

$$T^B + \sum_n V_B^{\rm ext}(\vec r_n) + \frac{1}{2}\sum_{i,j} V_B(r_{ij}) \;=\; T^A + \sum_n V_A^{\rm ext}(\vec r_n) + \frac{1}{2}\sum_{i,j} V_A(r_{ij})$$

$$\Rightarrow T + \sum_n V^{\rm ext}(\vec r_n) + \frac{1}{2}\sum_{i,j} V(r_{ij}) \;=\; {\rm const} \tag{1.47}$$

It is straightforward to identify the l.h.s. of this formula with the total energy of the system.

Note that if all $\vec r_{ij}$ do not change (e.g. for a rigid body), the last term in the above equation reduces to a constant which means that the internal forces do not do any work.

1.4 Central forces

A central force is a force directed towards a fixed point: $\vec F(\vec r) \;=\; \hat r F(r)$. Well-known examples: gravitational and Coulomb forces.

Central force is conservative:

$$\big(\vec\nabla \times \vec r f(r)\big)_i \;=\; \epsilon_{ijk}\frac{\partial}{\partial r_j} r_k f(r) \;=\; \epsilon_{ijk}\Big[\delta_{jk} + r_k r_j \frac{f'(r)}{r}\Big] \;=\; 0 \tag{1.48}$$

where I used chain rule $\frac{\partial r}{\partial r_j} = \frac{\partial r}{\partial r^2}\frac{\partial r^2}{\partial r_k} = \frac{r_k}{r} = \hat r_k$.

The potential for the central force depends on r. Indeed,

$$\vec F(\vec r) \;=\; -\vec\nabla V(r) \;=\; (\vec\nabla r)\frac{dV(r)}{dr} \;=\; \hat r V'(r) \;\;\Rightarrow\;\; F(r) \;=\; -V'(r) \tag{1.49}$$

1.4.1 Conservation laws

$$\text{Energy}: \qquad E \;=\; \frac{mv^2}{2} + V(r) \;=\; {\rm const} \tag{1.50}$$

$$\text{Angular momentum}: \qquad \vec\tau \;=\; \vec r \times \vec F \;=\; 0 \quad\Rightarrow\quad \vec L \;=\; \vec r \times \vec p \;=\; {\rm const} \tag{1.51}$$

Next, since $\vec L$ is conserved

$$\vec r\cdot\vec L \;=\; 0 \;\;\Rightarrow\;\; \frac{d}{dt}\vec r\cdot\vec L \;=\; 0 \;=\; \vec v\cdot\vec L + \vec r\cdot\frac{d}{dt}\vec L \;\;\Rightarrow\;\; \vec v\cdot\vec L \;=\; 0 \tag{1.52}$$

so the velocity is always orthogonal to $\vec L$ $\Rightarrow$
$\Rightarrow$ the motion occurs in a plane orthogonal to $\vec L$.

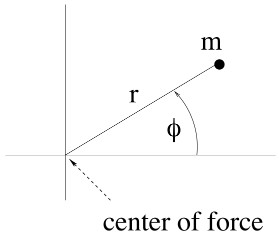

Figure 8. Polar coordinates in x, y plane.

1.4.2 Description of motion

It is convenient to assume $\vec{L} \parallel \hat{z}$ and use polar coordinates

$$x = r\cos\phi, \qquad y = r\sin\phi \tag{1.53}$$

In general, both r and ϕ change as particle moves:

$$\begin{aligned} v_x &= \dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ v_y &= \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \end{aligned} \tag{1.54}$$

The kinetic energy in polar coordinates takes the form

$$T = \frac{m}{2}(v_x^2+v_y^2) = \frac{m}{2}\big[(\dot{r}\cos\phi-r\dot{\phi}\sin\phi)^2+(\dot{r}\sin\phi+r\dot{\phi}\cos\phi)^2\big] = \frac{m}{2}\big[\dot{r}^2+r^2\dot{\phi}^2\big] \tag{1.55}$$

Since the energy is conserved

$$E = \frac{m}{2}(v_x^2+v_y^2)+V(r) = \frac{m}{2}\big(\dot{r}^2+r^2\dot{\phi}^2\big)+V(r) = \text{const} \tag{1.56}$$

Similarly, since angular momentum $\vec{L} = L\hat{z}$ is conserved

$$\begin{aligned} L_z &= (\vec{r}\times\vec{p})_z = xp_y - yp_x = m(x\dot{y}-y\dot{x}) \\ &= m\big[r\cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) - r\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)\big] = mr^2\dot{\phi} = \text{const} \end{aligned} \tag{1.57}$$

Note that sign of $\dot{\phi}$ is always the same as sign of L_z so no motion as in Fig. 9 is allowed. Note also that if ϕ increases r must decrease and *vice versa.* In addition, due to Eq. (1.57),

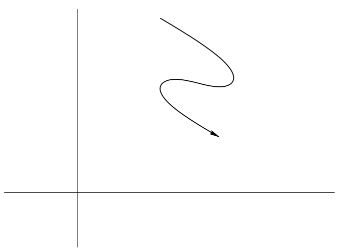

Figure 9. Such motion is not allowed (sign of $\dot{\phi}$ should not change)

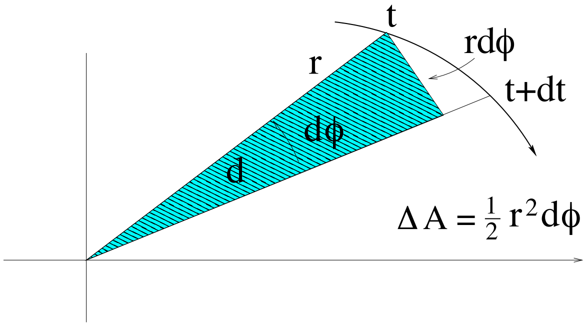

Figure 10. Area change

the change of the area swept by the particle is constant (see Fig. 10) Indeed,

$$\begin{aligned} dA &= \frac{1}{2}r(rd\phi) = \frac{1}{2}r^2\dot{\phi}dt \\ \Rightarrow \dot{A} &= \frac{1}{2}r^2\dot{\phi} = \frac{L}{2m} = \text{const} \end{aligned} \tag{1.58}$$

Kepler's 2nd Law: the rate of change of the area swept by the particle is constant

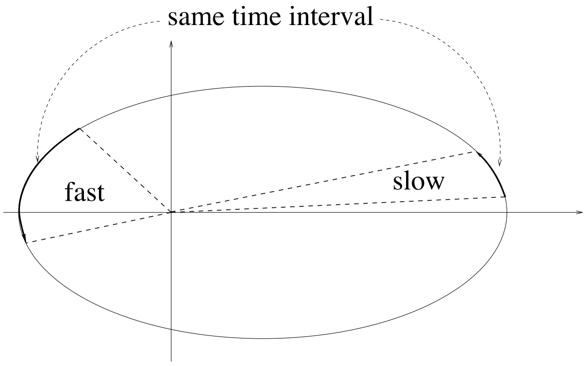

Figure 11. 2nd Kepler's law

1.4.3 Effective potential

Due to the conservation of angular momentum in the form of 2nd Kepler's law (1.58) the central problem can be reduced to 1-dimensional problem with an "effective potential":

$$\begin{aligned} E &= \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\phi}^2 + V(r) = \frac{m}{2}\dot{r}^2 + V(r) + \frac{m}{2}r^2\frac{L^2}{m^2r^4} \\ &= \frac{m}{2}\dot{r}^2 + V(r) + \frac{L^2}{2mr^2} = \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r) \end{aligned} \tag{1.59}$$

Thus, the energy of the particle in central potential is equal to the energy of the particle moving in one dimension (at $r > 0$) in the effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2} \tag{1.60}$$

Since $E - V_{\rm eff}(r) = \frac{m}{2}\dot{r}^2 \geq 0$, the equation

$$V(r) + \frac{L^2}{2mr^2} \leq E \tag{1.61}$$

determines the region of space where the motion can occur.

For example, consider gravitational force:

$$V(r) = -m\frac{\gamma}{r} \quad \Rightarrow \quad V_{\rm eff}(r) = \frac{L^2}{2mr^2} - \frac{m\gamma}{r} \tag{1.62}$$

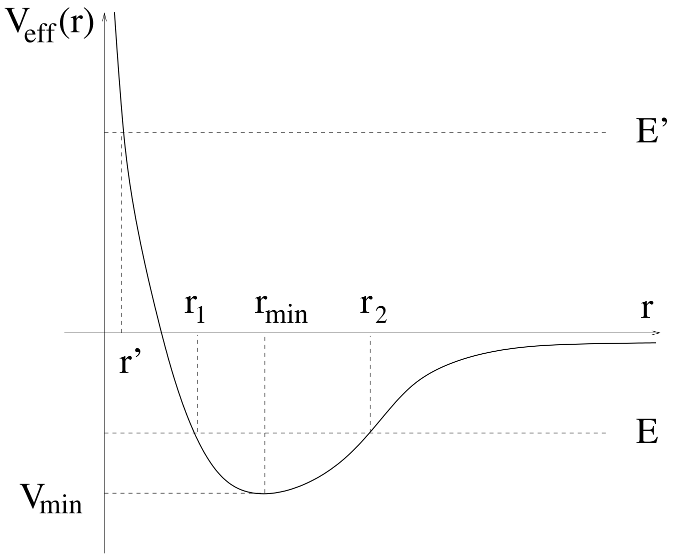

Figure 12. Effective potential for gravitational force

General considerations:

- $E < V_{\min}$: no motion is possible ($v^2 \not< 0!$)
- $V_{\min} \le E < 0$: motion is confined in the region of space between r_1 and r_2 (see Fig. 13)

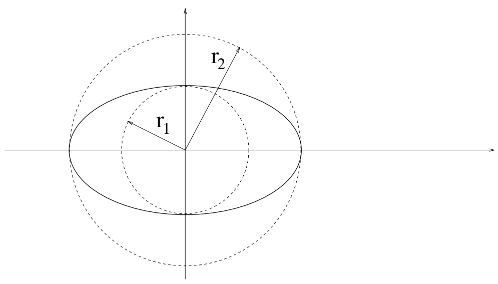

Figure 13. Motion at $E < 0$

- $E \ge 0$: motion in the region $r \ge r'$ (see Fig. 14)

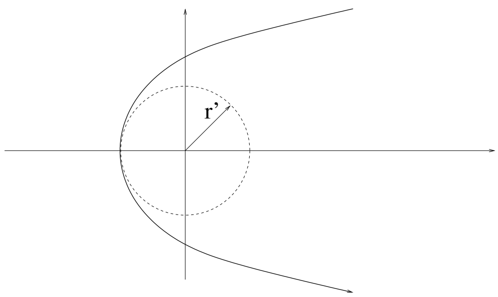

Figure 14. Motion at $E < 0$

Part III

Another example: harmonic oscillator in 3 dimensions. The effective potential is

$$V_{\text{eff}}(r) \;=\; \frac{m}{2}\omega^2 r^2 + \frac{L^2}{2mr^2} \tag{1.63}$$

(see Fig. 15)

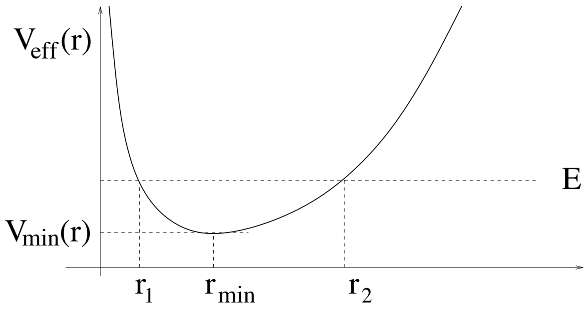

Figure 15. Effective potential for 3d oscillator

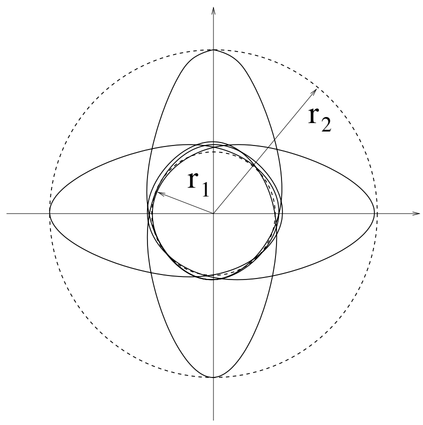

Figure 16. Typical trajectory for the 3d oscillator

We see that for any $E \geq V_{\text{min}}$ we can have motion only between r_1 and r_2.

1.4.4 Form of the trajectory in space

Motion (= trajectory in space) is uniquely specified by initial position and velocity. Indeed, the motion is determined by Newton's 2nd law

$$m\ddot{x} \; = \; F_x(x,y), \qquad m\ddot{y} \; = \; F_y(x,y) \tag{1.64}$$

and the solution of 2nd order differential equation is uniquely specified by initial conditions at $t = t_0$

$$\begin{aligned} x(t_0) \; &= \; x_0, & y(t_0) \; &= \; y_0 \\ \frac{dx(t)}{dt}\Big|_{t=t_0} \; &= \; \dot{x}(t_0) \; \equiv \; \dot{x}_0, & \frac{dy(t)}{dt}\Big|_{t=t_0} \; &= \; \dot{x}(t_0) \; \equiv \; \dot{y}_0 \end{aligned} \tag{1.65}$$

Alternatively, we can use four initial conditions

$$r_0 \; = \; r(t_0), \qquad \phi_0 \; = \; \phi(t_0), \qquad E \; = \; \frac{m}{2}(\dot{r}_0^2 + r_0^2\dot{\phi}_0^2) + V(r_0), \qquad L \; = \; mr_0^2\dot{\phi}_0. \tag{1.66}$$

Trajectory:
from Eq. (1.59) we get

$$\dot{r} \; = \; \frac{dr}{dt} \; = \; \pm\sqrt{\frac{2}{m}}\sqrt{E - V_{\text{eff}}(r)} \tag{1.67}$$

where the sign depends on whether $r(t)$ is increasing (sign "+") or decreasing (sign "-") at time t, In other words, the sign depends on the direction of radial motion (sign "+" for the motion out and sign "-" for the motion in). The trajectory will be the same: change of sign corresponds to change $t \to -t$ which does not alter the trajectory ("T-invariance of classical mechanics). Taking "+" solution we get

$$\begin{aligned} dt \; &= \; \sqrt{\frac{m}{2}}\frac{dr}{\sqrt{E - V_{\text{eff}}(r)}} \\ \Rightarrow t \; &= \; \int_0^t dt \; = \; \sqrt{\frac{m}{2}}\int_{r_0}^r dr' \frac{1}{\sqrt{E - V_{\text{eff}}(r')}} \end{aligned} \tag{1.68}$$

Once the integration is performed, the above equation can be inverted to provide $r = r(t)$. Moreover, if $r(t)$ is known, the equation (1.57) can be easily integrated

$$\frac{d\phi}{dt} \; = \; \frac{L}{mr^2(t)} \qquad \Rightarrow \qquad \phi(t) - \phi_0 \; = \; \int_{\phi_0}^{\phi} d\phi' \; = \; \frac{L}{m}\int_0^t dt' \frac{1}{r^2(t')} \tag{1.69}$$

The equations (1.68) and (1.69) give the trajectory in the parametric form $r = r(t)$ and $\phi = \phi(t)$. One can eliminate t from these equations and get the trajectory in polar coordinates in the form $r = r(\phi)$. There is , however, a more direct way to determine $r(\phi)$. From Eqs. (1.57) and (1.68) we see that

$$\begin{aligned} dt \; &= \; \frac{mr^2}{L}d\phi \;\; \Rightarrow \;\; \pm\sqrt{\frac{m}{2}}\frac{dr}{\sqrt{E - V_{\text{eff}}(r)}} \; = \; \frac{mr^2}{L}d\phi \\ \Rightarrow \; d\phi \; &= \; \pm\frac{L}{\sqrt{2m}}\frac{dr}{r^2\sqrt{E - V_{\text{eff}}(r)}} \\ \Rightarrow \; \phi - \phi_0 \; &= \; \int_{\phi_0}^{\phi} d\phi' \; = \; \pm\frac{L}{\sqrt{2m}}\int_{r_0}^{r} dr' \frac{1}{r'^2\sqrt{E - V_{\text{eff}}(r')}} \end{aligned} \tag{1.70}$$

Suppose that at $t = 0$ the particle is at one of the inversion points, say r_1 (see Fig. 13). Let us calculate the change in ϕ as the particle moves to outer inversion point r_2 and back. We get

$$\Delta\phi \;=\; 2(\phi_2 - \phi_1) \;=\; \sqrt{\frac{2L}{m}}\int_{r_1}^{r_2} dr' \frac{1}{r'^2\sqrt{E - V_{\text{eff}}(r')}} \tag{1.71}$$

Q: When the orbit is closed?
A: when $\Delta\phi = 2\pi\frac{m}{n}$. Indeed, after n repetitions of time interval required to get from r_1 to

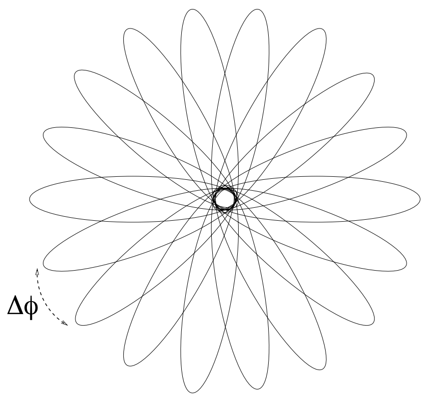

Figure 17. Typical trajectory for the confined motion

r_2 and back to r_1, the position vector, having done m round, returns to the initial point (see Fig. 17).

In general, confined motion is not closed and after sufficiently large time the trajectory will come infinitely close to any point between r_1 and r_2.

There are two cases which lead to closed orbits: $V(r) \sim \frac{1}{r}$ and $V(r) \sim r^2$.

For the potential $V(r) = \frac{\gamma m}{r}$ (inverse square gravitational force) we have

$$V(r) \;=\; -\frac{\gamma m}{r} \quad\Rightarrow\quad V_{\text{eff}} \;=\; -\gamma\frac{m}{r} + \frac{L^2}{2mr^2} \tag{1.72}$$

so we can find r_{min} and V_{min}:

$$V'_{\text{eff}}(r) \;=\; 0 \quad\Rightarrow \frac{m\gamma}{r_{\text{min}}^2} \;=\; \frac{L^2}{mr_{\text{min}}^3} \quad\Rightarrow\quad r_{\text{min}} \;=\; \frac{L^2}{\gamma m^2}, \quad V_{\text{min}} \;=\; -\frac{m^3\gamma^2}{2L^2} \tag{1.73}$$

1.4.5 Confined motion in a gravitational field

If $V_{\text{min}} \le E < 0$ the motion is confined, see Fig. 12. If $E = V_{\text{min}}$, the motion is circular since due to Eq. (1.59) $\dot{r} = 0 \;\Rightarrow\; r = \text{const}$. If $E > V_{\text{min}}$ (but $E < 0$) the trajectory of the particle is elliptical.

Let us demonstrate this. From Eq. (1.70)

$$\phi - \phi_0 \;=\; \int_{\phi_0}^{\phi} d\phi' \;=\; -\frac{L}{\sqrt{2m}}\int_{r_0}^{r} dr' \frac{1}{r'^2\sqrt{E + \gamma\frac{m}{r'} - \frac{L^2}{2mr'^2}}} \tag{1.74}$$

(here we took "-" sign which differs from "+ sign" by $t \to -t$ substitution). After change of variables

$$u' = \frac{1}{r'} \quad \Rightarrow \quad du' = -\frac{dr'}{r'^2} \tag{1.75}$$

the integral (1.74) reduces to

$$\phi - \phi_0 = \frac{L}{\sqrt{2m}} \int_{u_0}^{u} du' \frac{1}{\sqrt{E + \gamma m u' - \frac{L^2 u'^2}{2m}}} = \int_{u_0}^{u} du' \frac{1}{\sqrt{-2m\frac{|E|}{L^2} + \left(\frac{m^2\gamma}{L^2}\right)^2 - \left(\frac{m^2\gamma}{L^2} - u'\right)^2}}$$
$$= \int_{u_0}^{u} du' \frac{1}{\sqrt{a^2 - \left(\frac{m^2\gamma}{L^2} - u'\right)^2}}, \qquad a^2 \equiv -2m\frac{|E|}{L^2} + \left(\frac{m^2\gamma}{L^2}\right)^2 = \frac{2m}{L^2}(E - V_{\min}) \tag{1.76}$$

Another change of variables:

$$x = \frac{u_* - u'}{a} \quad \Rightarrow \quad dx = -\frac{1}{a} du' \qquad \text{where} \quad u_* = \frac{1}{r_{\min}} = \frac{m^2\gamma}{L^2} \tag{1.77}$$

We get

$$\phi - \phi_0 = -\int_{\frac{u_* - u_0}{a}}^{\frac{u - u_0}{a}} dx \frac{1}{\sqrt{1 - x^2}} = \arccos x \Big|_{\frac{u_* - u_0}{a}}^{\frac{u - u_0}{a}} \tag{1.78}$$
$$\Rightarrow \phi = \phi_0 + \arccos\frac{u_* - u}{a} - \arccos\frac{u_* - u_0}{a} \quad \Rightarrow \quad \cos(\phi - \phi_0 + \delta) = \frac{u_* - u}{a}$$

where $\delta = \arccos\frac{u_* - u_0}{a}$.

In terms of original variables

$$\frac{u_* - u}{a} = \left(\frac{\gamma m^2}{L^2} - u\right) \frac{1}{\sqrt{\frac{m^4\gamma^2}{L^4} - \frac{2m|E|}{L^2}}} = \left(1 - \frac{L^2 u}{\gamma m^2}\right) \frac{1}{\sqrt{1 - \frac{2|E|L^2}{m^3\gamma^2}}} \tag{1.79}$$

so the trajectory equation (1.76) takes the form

$$\left(1 - \frac{L^2 u}{\gamma m^2}\right) \frac{1}{\sqrt{1 - \frac{2|E|L^2}{m^3\gamma^2}}} = \cos(\phi - \phi_0 + \delta) \tag{1.80}$$
$$\Rightarrow \frac{1}{r} = \frac{m^2\gamma}{L^2}\left[1 - \sqrt{1 - \frac{2|E|L^2}{m^3\gamma^2}} \cos(\phi - \phi_0 + \delta)\right] = C\left[1 - \epsilon\cos(\phi - \phi_0 + \delta)\right]$$

where $C = \frac{m^2\gamma}{L^2} = \frac{1}{r_{\min}}$ and $\epsilon = \sqrt{1 - \frac{|E|}{|V_{\min}|}} < 1$.

Without loss of generality, let us assume that $\phi_0 = 0$ at $t = 0$ so the equation (1.76) takes the form

$$\frac{1}{r} = C(1 - \epsilon\cos\phi) \tag{1.81}$$

This is actually an equation for ellipse

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.82}$$

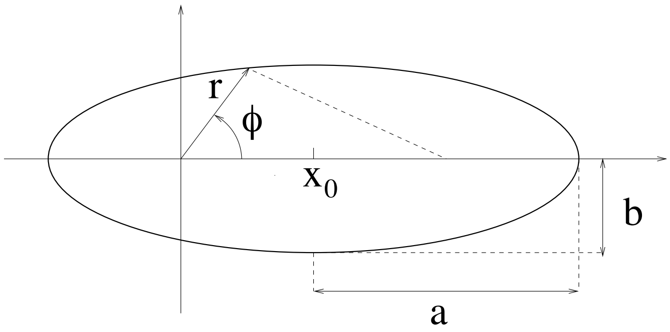

Figure 18. Elliptical trajectory in gravitational potential

with

$$x_0 \;=\; \frac{\epsilon}{C(1-\epsilon^2)}, \qquad a \;=\; \frac{1}{C(1-\epsilon^2)}, \qquad b \;=\; \frac{1}{C\sqrt{1-\epsilon^2}} \tag{1.83}$$

The center of force (origin $\vec{r}=0$ is located in one of the foci of the ellipse. Indeed, let us check

$$\begin{aligned} r+\sqrt{r^2+4x_0^2-4x_0 r\cos\phi} \;&=\; r+\sqrt{r^2+4x_0^2-4x_0\big(\frac{r}{\epsilon}-\frac{1}{C\epsilon}\big)} \\ =\; r+\big|r-\frac{2}{C(1-\epsilon^2)}\big| \;&=\; \frac{2}{C(1-\epsilon^2)} \;=\; 2a \end{aligned} \tag{1.84}$$

Thus, the trajectory of the confined motion in a gravitational force is an ellipse with the focus being the center of the force. The parameters of the ellipse are

$$a \;=\; \frac{1}{C(1-\epsilon^2)} \;=\; \frac{m\gamma}{2|E|}, \qquad b \;=\; \frac{L}{\sqrt{2m|E|}}, \qquad x_0 \;=\; \frac{m\gamma}{2|E|}\sqrt{1-\frac{2|E|L^2}{m^3\gamma^2}} \tag{1.85}$$

(Recall that $|E| < V_{\rm min} \;=\; \frac{m^3\gamma^2}{2L^2}$, see Eq. (1.73)).

Kepler's 3rd law.

In a period T the particle sweeps the area of the ellipse

$$A \;=\; 4b\int_0^a dx\sqrt{1-\frac{x^2}{a^2}} \;=\; 4ab\int_0^1 dx\sqrt{1-x^2} \;=\; 2ab\int_0^1 dy\, y^{\frac{-1}{2}}(1-y)^{\frac{1}{2}} \;=\; \pi ab \;=\; \pi a^2\sqrt{1-\epsilon^2} \tag{1.86}$$

On the other hand,

$$A \;=\; \int_0^T dt\;\dot{A} \;=\; T\frac{L}{2m} \quad\Rightarrow\quad T \;=\; \frac{2m}{L}\pi a^2\sqrt{1-\epsilon^2} \;=\; 2\pi\frac{a^{\frac{3}{2}}}{\sqrt{\gamma}} \tag{1.87}$$

This is Kepler's 3rd law: $T^2 \sim a^3$.

Similar methods can be applied to other potentials. In general, confined orbits (if they exist) are open. The only known potentials for which all confined orbits are closed, are gravitational (or Coulomb) potentials $V(r)\sim\frac{1}{r}$ and harmonic potentials $V(r)\sim r^2$.

1.4.6 Open motion in the gravitational field.

Open motion corresponds to $E \geq 0$ (see Fig. 12). The solution of the Newton's equations proceeds in a way similar to the case $E < 0$ even thogth the resulting orbits are very different (parabola for $E = 0$ and hyperbola for $E > 0$). The equation for trajectory is still (1.74) but now we have $E \geq 0$.

$$\phi - \phi_0 \;=\; \int_{\phi_0}^{\phi} d\phi' \;=\; -\frac{L}{\sqrt{2m}} \int_{r_0}^{r} dr' \frac{1}{r'^2 \sqrt{E + \gamma \frac{m}{r'} - \frac{L^2}{2mr'^2}}} \tag{1.88}$$

Next, we make the same substitutions $u' = \frac{1}{r'}$ and $x = \frac{u_* - u'}{a}$ with $u_* = (\frac{m^2\gamma}{L^2} = \frac{1}{r_{\text{min}}}$ and $a = \sqrt{\frac{m^4\gamma^2}{L^4} + \frac{2mE}{L^2}}$ (see Eqs. (1.75-1.77)) and get

$$\begin{aligned} \phi - \phi_0 \;&=\; -\int_{\frac{u_* - u_0}{a}}^{\frac{u - u_0}{a}} dx \frac{1}{\sqrt{1 - x^2}} \;=\; \arccos x \Big|_{\frac{u_* - u_0}{a}}^{\frac{u - u_0}{a}} \qquad (1.89) \\ \Rightarrow \;\phi \;&=\; \phi_0 + \arccos \frac{u_* - u}{a} - \arccos \frac{u_* - u_0}{a} \;\;\Rightarrow\;\; \cos(\phi - \phi_0 + \delta) \;=\; \frac{u_* - u}{a} \end{aligned}$$

where $\delta \;=\; \arccos \frac{u_* - u_0}{a}$.

In terms of original variables

$$\frac{u_* - u}{a} \;=\; (\frac{\gamma m^2}{L^2} - u) \frac{1}{\sqrt{\frac{m^4\gamma^2}{L^4} + \frac{2mE}{L^2}}} \;=\; (1 - \frac{L^2 u}{\gamma m^2}) \frac{1}{\sqrt{1 + \frac{2EL^2}{m^3\gamma^2}}} \tag{1.90}$$

so the trajectory equation (1.76) takes the form

$$\begin{aligned} (1 - \frac{L^2 u}{\gamma m^2}) \frac{1}{\sqrt{1 - \frac{2|E|L^2}{m^3\gamma^2}}} \;&=\; \cos(\phi - \phi_0 + \delta) \qquad (1.91) \\ \Rightarrow \frac{1}{r} \;&=\; \frac{m^2\gamma}{L^2} \Bigg[1 - \sqrt{1 + \frac{2EL^2}{m^3\gamma^2}} \cos(\phi - \phi_0 + \delta)\Bigg] \;=\; C\big[1 - \epsilon \cos(\phi - \phi_0 + \delta)\big] \end{aligned}$$

where $C = \frac{m^2\gamma}{L^2} = \frac{1}{r_{\text{min}}}$ and $\epsilon = \sqrt{1 + \frac{E}{|V_{\text{min}}|}} \geq 1$.

Again, w.l.o.g. we assume that $\phi_0 - \delta = 0$ at $t = 0$ so the equation (1.76) takes the form

$$\frac{1}{r} \;=\; C(1 - \epsilon \cos \phi) \tag{1.92}$$

with $\epsilon \geq 1$. Actually, the case $E > 0$ corresponds to the hyperbola trajectory while the trajectory of the particle with $E = 0$ is parabolic.

Let us start with the first case $E > 0$. If we continue analytically the equation for ellipse (1.82)

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} \;=\; 1 \;=\; C^2(1 - \epsilon^2)^2 \Big(x - \frac{\epsilon}{C(1 - \epsilon^2)}\Big)^2 + y^2 C^2 (1 - \epsilon^2) \tag{1.93}$$

to $\epsilon > 1$, we get

$$C^2(1 - \epsilon^2)^2 \Big(x + \frac{\epsilon}{C(\epsilon^2 - 1)}\Big)^2 \;=\; 1 + y^2 C^2 (\epsilon^2 - 1) \tag{1.94}$$

which can be rewritten as

$$\frac{(x+x_0)^2}{a^2} - \frac{y^2}{b^2} \;=\; 1 \tag{1.95}$$

with

$$a \;=\; \frac{1}{C(\epsilon^2-1)} \;=\; \frac{m\gamma}{2E}, \quad b \;=\; \frac{1}{C\sqrt{\epsilon^2-1}} \;=\; \frac{L}{\sqrt{2mE}}, \quad x_0 \;=\; \frac{\epsilon}{C(\epsilon^2-1)} \;=\; \frac{m\gamma}{2E}\sqrt{1+\frac{2EL^2}{m^3\gamma^2}} \tag{1.96}$$

The Eq. (1.96) is an equation of a hyperbola. The asymptotic behavior at large x, y is

$$\pm\frac{y}{b} \;=\; \pm\sqrt{\frac{(x+x_0)^2}{a^2}-1} \simeq \pm\frac{x+x_0}{a} \quad\Rightarrow\quad y \;=\; \pm\frac{b}{a}(x+x_0) \tag{1.97}$$

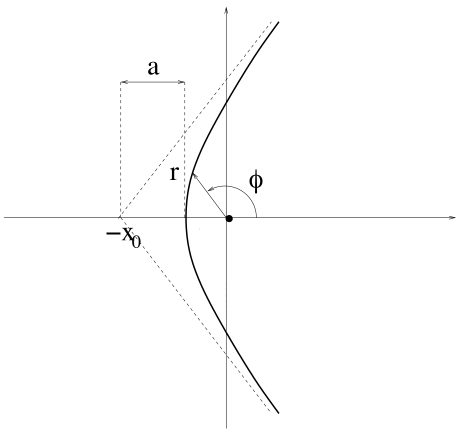

Figure 19. Hyperbolic trajectory in gravitational potential at $E > 0$

Case E=0 (parabolic motion)
The equation (1.100) turns to

$$\frac{1}{r} \;=\; C(1-\cos\phi) \tag{1.98}$$

where $C \;=\; \frac{1}{r_{\min}} \;=\; \frac{m^2\gamma}{L^2}$.
which can be rewritten as an equation for a parabola:

$$\frac{1}{r} \;=\; C(1-\cos\phi) \;\Rightarrow\; \frac{1}{r} \;=\; C - C\frac{x}{r} \;\Rightarrow\; 1+Cx = Cr \;\Rightarrow\; 1+2Cx = C^2y^2 \tag{1.99}$$

or

$$y^2 \;=\; \frac{2}{C}\Big(x+\frac{1}{2C}\Big) \;=\; 2r_{\min}\Big(x+\frac{r_{\min}}{2}\Big) \tag{1.100}$$

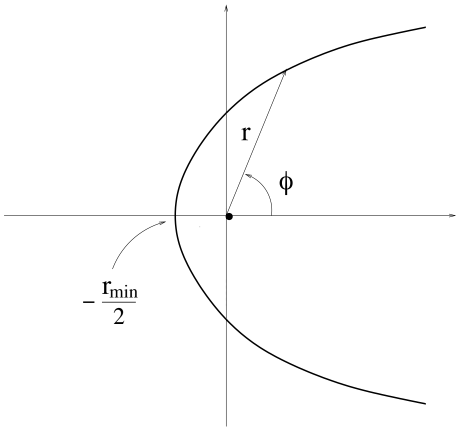

Figure 20. Parabolic trajectory in gravitational potential at $E = 0$

1.5 Two-body problem with central potential

Two particles: m_1 and m_2, potential $V = V(r)$ where $r \equiv |\vec{r}_1 - \vec{r}_2|$. Newton's 2nd law:

$$m_1\ddot{\vec{r}}_1 = -\vec{\nabla}_1 V(r) = -\hat{r}\frac{dV(r)}{dr} \tag{1.101}$$

$$m_1\ddot{\vec{r}}_2 = -\vec{\nabla}_2 V(r) = \hat{r}\frac{dV(r)}{dr} \tag{1.102}$$

CM and relative coordinates

$$\begin{aligned} \vec{R} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned} \tag{1.103}$$

Inverse formulas read

$$\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{m_1+m_2}\vec{r} \quad \Rightarrow \quad \ddot{\vec{r}}_1 = \ddot{\vec{R}} + \frac{m_2}{m_1+m_2}\ddot{\vec{r}} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{m_1+m_2}\vec{r} \quad \Rightarrow \quad \ddot{\vec{r}}_2 = \ddot{\vec{R}} - \frac{m_1}{m_1+m_2}\ddot{\vec{r}} \end{aligned} \tag{1.104}$$

Adding Eqs. (2.1) and (1.102) one gets

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0 \quad \Leftrightarrow \quad (m_1+m_2)\ddot{\vec{R}} = 0 \quad \Rightarrow \quad \ddot{\vec{R}} = 0 \tag{1.105}$$

so the center of mass moves along straight line (or remains at rest). As to the relative separation, subtracting Eq. (1.102) from Eq. (2.1) we get

$$\begin{aligned} m_1\ddot{\vec{r}}_1 - m_2\ddot{\vec{r}}_2 &= -2\vec{r}\frac{dV(r)}{dr} \\ \Rightarrow \frac{m_1m_2}{m_1+m_2}\ddot{\vec{r}} &= -\vec{r}\frac{dV(r)}{dr} \end{aligned} \tag{1.106}$$

where

$$\mu \equiv \frac{m_1m_2}{m_1+m_2} \tag{1.107}$$

So, the two-body problem with a potential depending on the separation reduces to a one-body problem of mass μ moving in a central potential $V(r)$

Part IV

1.6 Scattering

Consider motion of a particle in central potential $V(r)$ which we assume to vanish at infinity $V(r) \overset{r\to\infty}{\to} 0$. The energy of a free motion at $t \to -\infty$ is $E = \frac{m}{2}v_\infty^2$ and the angular momentum is $L = mv_\infty b$ where b is called an impact factor. The typical picture of the scattering from a repulsive potential is shown in Fig.

One can have in mind Coulomb potential $V(r) = \frac{qQ}{4\pi r}$ as a typical example.

The point at the minimal distance r_0 is the inversion point for given energy E and angular momentum L. Since $\dot{r}(r_0) = 0$ from Eq. (1.59) we see that r_0 is a solution of the equation

$$V_{\rm eff}(r_0) \;=\; E \qquad \Leftrightarrow \qquad V(r_0) + \frac{L^2}{2mr_0^2} \;=\; E \tag{1.108}$$

If we know r_0, the angle ϕ_0 can be obtained from Eq. (1.70)

$$\phi_0 \;=\; -\frac{L}{\sqrt{2m}}\int_\infty^{r_0} dr' \frac{1}{r'^2\sqrt{E - V(r') - \frac{L^2}{2mr'^2}}} \tag{1.109}$$

(The minus sign is due to the fact that $\dot{r} < 0$ if the particle is approaching the scattering center).

After reaching r_0 the particle moves again to infinity and the change of angle between r_0 and infinity is

$$\phi_0' \;=\; \frac{L}{\sqrt{2m}}\int_{r_0}^{\infty} dr' \frac{1}{r'^2\sqrt{E - V(r') - \frac{L^2}{2mr'^2}}} \tag{1.110}$$

Note that $\phi_0 = \phi_0'$ and the trajectory is symmetric with respect to line parallel to vector $\vec{r}_0$ (see Fig. 21)

For future use, it is convenient to represent ϕ_0 in terms of b and v_∞ as

$$\phi_0 \;=\; \int_{r_0}^{\infty} dr' \frac{b}{r'^2\sqrt{1 - \frac{b}{r'^2} - \frac{V(r')}{mv_\infty^2/2}}} \tag{1.111}$$

The deflection angle (the angle between velocities at plus and minus infinity) is

$$\theta \;=\; \pi - 2\phi_0 \tag{1.112}$$

Q: What changes if the potential changes to attractive $V(r) \to -V(r)$ (for example $\frac{qQ}{4\pi r} \to -\frac{qQ}{4\pi r}$)?
A: Very little: formula (1.111) stays the same but the reflection angle is now $\theta = 2\phi_0 - \pi$, see Fig. 22. In fact, we can treat two cases (repulsive and attractive potentials) similarly just using $\theta = |2\phi_0 - \pi|$.

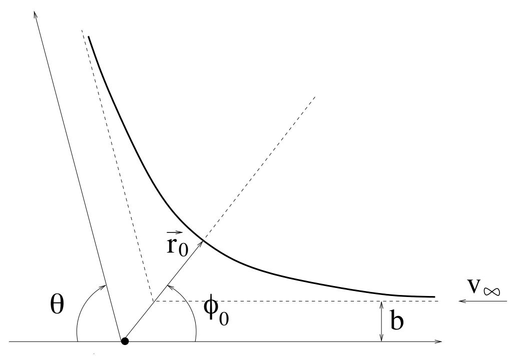

Figure 21. Scattering from a repulsive potential

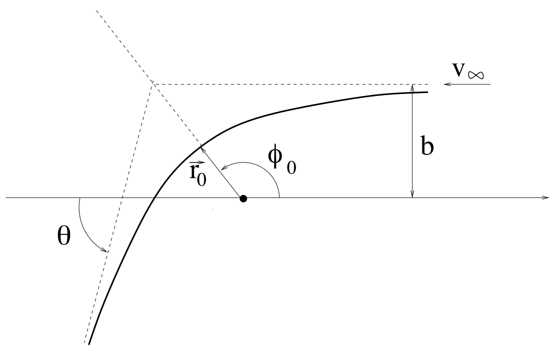

Figure 22. Scattering from an attractive potential

1.6.1 Cross section

Consider a uniform beam of particles incident on a central potential

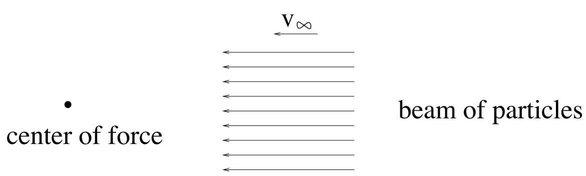

Figure 23. A beam of particles incident on a central potential

Flux Φ is a number of particles per unit area per unit time

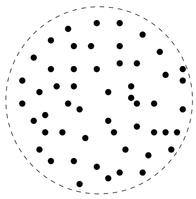

Figure 24. Transverse view of a beam of particles

Each particle has a definite b abd v_∞ and will be deflected by angle $\theta = |\pi - 2\phi_0|$. Let us consider now particles in a ring between b and $b + \Delta b$. The number of particles crossing area of a ring $b < r < b + \Delta b$ per unit time is

$$dn \;=\; 2\pi b \Delta b\ \Phi \tag{1.113}$$

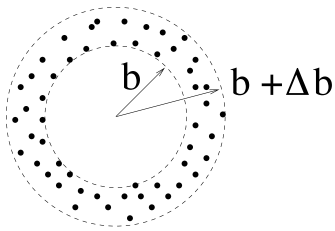

Figure 25. Particles in a ring between b and $b + \Delta b$

These particles will be deflected by angle between θ and $\theta + \Delta\theta$, see Fig. 26. (Due to azimuthal symmetry, the deflection angle $\Delta\theta$ does not depend on ϕ).

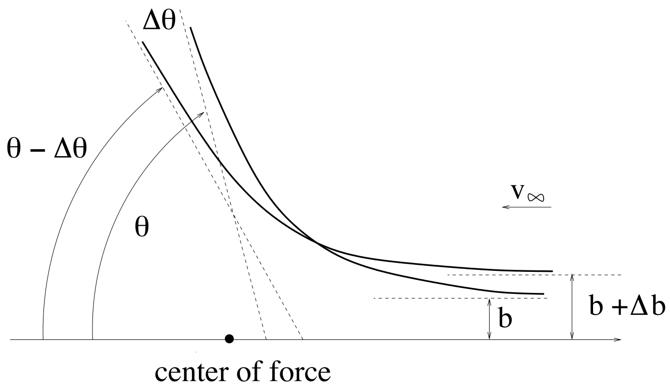

Figure 26. Scattering of particles with impact parameter between b and $b + \Delta b$

Cross section $d\sigma$ is defined as

$$dn(\theta) \;=\; \Phi d\sigma(\theta) \tag{1.114}$$

Note that $d\sigma$ has the dimension of an area since dn has a dimension of $\frac{1}{\text{time}}$ (from Eq. (1.113) $dn = \frac{\text{number of particles}}{\text{time}}$).

Since $\theta = |\pi - 2\phi_0(b)|$ one may think of b as a function of θ and get from Eqs. (1.113) and (1.114)

$$\not{\Phi} d\sigma(\theta) = \not{\Phi} 2\pi b db \quad \Rightarrow \quad d\sigma(\theta) = 2\pi b \left|\frac{db(\theta)}{d\theta}\right| d\theta \tag{1.115}$$

The reason for modulus $\left|\frac{db(\theta)}{d\theta}\right|$ in the r.h.s. of this equation is that $d\sigma(\theta)$ is a positive definite quantity $(=\frac{\text{number of particles}}{\text{flux}})$ while $b(\theta)$ is generally decreasing function of θ (the greater the impact parameter b, the smaller is the deflection angle θ), see Fig. 26.

It is convenient to write down the derivative of the cross section with respect to solid angle (so-called "differential cross section" $\frac{d\sigma}{d\Omega}$). Recall that $d\Omega \equiv \sin\theta d\theta d\phi \Rightarrow$

$$d\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db(\theta)}{d\theta}\right| d\Omega \quad \Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db(\theta)}{d\theta}\right| \tag{1.116}$$

The total cross section is defined as

$$\sigma_{\text{tot}} \equiv \int d\Omega \frac{d\sigma}{d\Omega} \tag{1.117}$$

so it is a number of particles scattered in a unit time in all directions divided by flux.

Example: scattering from a rigid ball of radius a. The potential is

$$V(r) = 0 \quad \text{if } r \geq a \quad \text{and} \quad V(r) = \infty \quad \text{if } r < a \tag{1.118}$$

From Fig. 27 we see that $\sin\phi_0 = \frac{b}{a}$ (for $b < a$, at $b \geq a$ the particle will not be deflected)

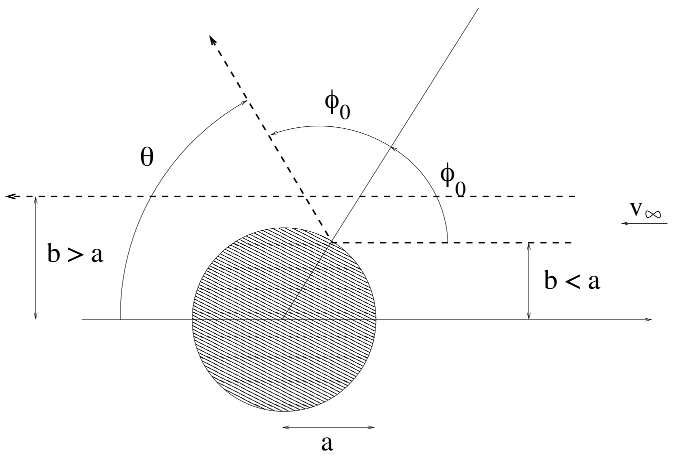

Figure 27. Scattering from the rigid ball

so

$$\theta = \pi - 2\arcsin\frac{b}{a} \quad \Rightarrow \quad \frac{b}{a} = \sin\frac{\pi - \theta}{2} = \cos\frac{\theta}{2} \quad \Rightarrow \quad \frac{db}{d\theta} = -\frac{a}{2}\sin\frac{\theta}{2} \tag{1.119}$$

and therefore

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\Big|\frac{db(\theta)}{d\theta}\Big| = \frac{a\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \times \frac{a}{2}\sin\frac{\theta}{2} = \frac{a^2}{4} \tag{1.120}$$

Not that the obtained cross section

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4} \tag{1.121}$$

is isotropic (does not depend on θ. In other words, regardless of where the detector is placed, it will detect the same number of particles per unit time per unit solid angle (for a given flux Φ).

The total cross section is

$$\sigma_{\rm tot} \equiv \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega\, \frac{a^2}{4} = 4\pi \times \frac{a^2}{4} = \pi a^2 \tag{1.122}$$

(which means that we defined the cross section (1.115) in accordance with our everyday intuition).

1.6.2 Rutherford scattering

Consider two particle with masses m and M and charges ze and Ze. In c.m. variables (1.103) the effective potential is

$$V_{\rm eff}(r) = \frac{Zze^2}{r} + \frac{L^2}{2\mu r^2}, \qquad \mu \equiv \frac{mM}{m+M} \tag{1.123}$$

(see Fig. 28)

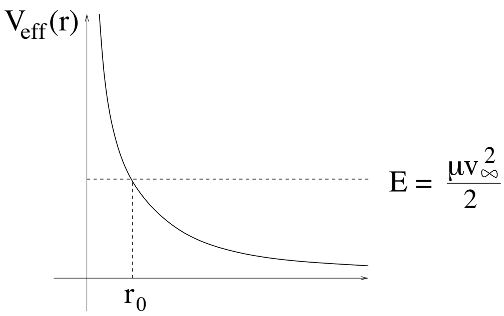

Figure 28. Effective potential for a scattering from a Coulomb center

The inversion point r_0 can be found from the equation

$$E = \frac{Zze^2}{r_0} + \frac{L^2}{2\mu r_0^2}, \qquad \mu \equiv \frac{mM}{m+M} \tag{1.124}$$

or, in terms of v_∞ and b

$$2\alpha\frac{b}{r_0} + \Big(\frac{b}{r_0}\Big)^2 = 1, \qquad \alpha \equiv \frac{Zze^2}{\mu v_\infty b} \tag{1.125}$$

This is a quadratic equation with a (positive) solution

$$r_0 \;=\; \frac{b}{\sqrt{1+\alpha^2}-\alpha} \tag{1.126}$$

Now we can find the angle ϕ_0. Since we are considering repulsive force ($Zz > 0$) the trajectory looks like Fig. 29

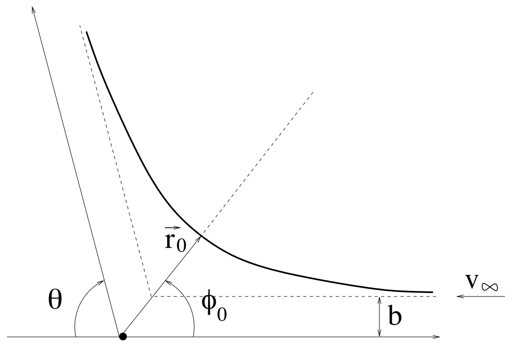

Figure 29. Scattering of particles from a Coulomb center

and therefore $\theta \;=\; \pi - 2\phi_0$ where ϕ_0 is given by Eq. (1.111)

$$\phi_0 \;=\; \int_{r_0}^{\infty} dr' \frac{b}{r'^2\sqrt{1-\frac{b}{r'^2}-\frac{2Zze^2}{\mu v_\infty^2 r'}}} \;\overset{u'=1/r'}{=}\; \int_0^{\frac{1}{r_0}} \frac{du'}{\sqrt{1-b^2u'^2-2\alpha b u'}} \tag{1.127}$$

$$\overset{x=u'b}{=} \int_0^{\frac{b}{r_0}} \frac{dx}{\sqrt{1-x^2-2\alpha x}} \;=\; \arcsin\frac{x+\alpha}{\sqrt{1+\alpha^2}}\Big|_0^{b/r_0} \;=\; \frac{\pi}{2} - \arcsin\frac{\alpha}{\sqrt{1+\alpha^2}}$$

because $(\frac{b}{r_0}+\alpha)^2 \;=\; 1+\alpha^2$, see Eq. (1.126). The deflection angle takes the form

$$\theta \;=\; \pi - 2\phi_0 \;\Rightarrow\; 2\arcsin\frac{\alpha}{\sqrt{1+\alpha^2}} \;\Rightarrow\; \sin\frac{\theta}{2} \;=\; \frac{\alpha}{\sqrt{1+\alpha^2}} \tag{1.128}$$

and therefore

$$\frac{1}{\sin^2\frac{\theta}{2}} \;=\; 1+\frac{1}{\alpha^2} \;=\; 1+b^2\Big(\frac{\mu v_\infty^2}{Zze^2}\Big)^2 \tag{1.129}$$

To find differential cross section from Eq. (1.116) we need to rewrite the impact parameter b as a function of deviation angle θ which is easily done inverting the above equation: [2]

$$b(\theta) \;=\; \Big|\frac{Zze^2}{\mu v_\infty^2}\Big| \cot\frac{\theta}{2} \tag{1.130}$$

The differential cross section (1.116) takes the form

$$\frac{d\sigma}{d\Omega} \;=\; \frac{b}{\sin\theta}\Big|\frac{db(\theta)}{d\theta}\Big| \;=\; \Big|\frac{zZe^2}{2\mu v_\infty^2}\Big|^2 \frac{1}{\sin^4\frac{\theta}{2}} \tag{1.131}$$

This is the famous Rutherford's formula. Properties:

[2] We have derived this formula for the repulsive potential, but it can be easily demonstrated thatEq. (1.130) equation is valid for attractive Coulomb potential as well.

- $\frac{d\sigma}{d\Omega}$ is independent of the sign of charges ze and Ze ($\equiv$ cross section is the same for attractive and repulsive Coulomb potential).
- $\frac{d\sigma}{d\Omega} \sim \frac{1}{\theta^4}$ for small angles (large impact parameters) $\Rightarrow$
- The integral for the total cross section (1.117) $\sigma_{\rm tot} = \int d\Omega \frac{d\sigma}{d\Omega}$ diverges at small θ

The last property means that the total cross section $\sigma_{\rm tot}$ is poorly defined for Coulomb potential since all particles are deflected regardless of how large is b. This behavior (divergence of $\sigma_{\rm tot}$) is a characteristic of potentials falling as $\frac{1}{r}$ at large separations.

Part V

2 Accelerated coordinate systems

2.1 Rotating coordinate systems

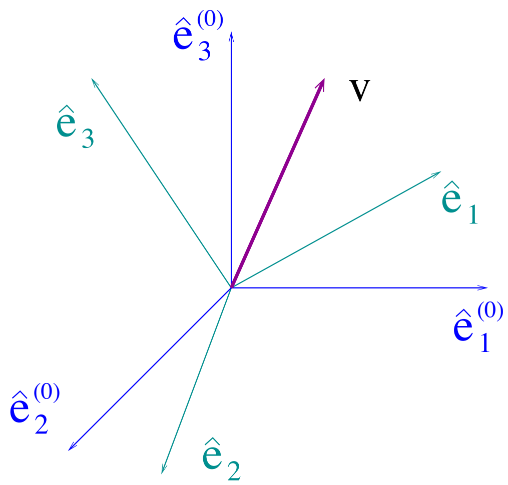

Figure 30. Transformation to a rotating coordinate system

$$\begin{aligned}\vec{v} &= v_i^{(0)}\hat{e}_i^{(0)} \\ \vec{v} &= v_i\hat{e}_i \end{aligned} \tag{2.1}$$

Since unit vectors $e_i^{(0)}$ are fixed

$$\left(\frac{d\vec{v}}{dt}\right)_{\rm inertial} = \frac{dv_i^{(0)}}{dt}\hat{e}_i^{(0)} \tag{2.2}$$

and therefore

$$\left(\frac{d\vec{v}}{dt}\right)_{\rm inertial} = \frac{dv_i}{dt}\hat{e}_i + v_i\frac{d\hat{e}_i}{dt} \tag{2.3}$$

The first term in the r.h.s. is the rate of change of $\vec{v}$ as seen by an observer in the moving (body-fixed) frame

$$\left(\frac{d\vec{v}}{dt}\right)_{\text{body}} = \frac{dv_i}{dt}\hat{e}_i \tag{2.4}$$

so

$$\left(\frac{d\vec{v}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{v}}{dt}\right)_{\text{body}} + v_i\frac{d\hat{e}_i}{dt} \tag{2.5}$$

2.2 Infinitesimal rotations

Suppose vectors $\hat{e}_i$ are changing in time: at $t+dt$ we have $\hat{e}_i(t+dt) = \hat{e}_i(t) + d\hat{e}_i$. Since $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ at any time t we get

$$\hat{e}(t+dt)\cdot\hat{e}(t+dt) = \hat{e}(t)\cdot\hat{e}(t) + 2\hat{e}(t)\cdot d\hat{e} + O(dt^2) \Rightarrow \hat{e}\cdot d\hat{e} = 0 \tag{2.6}$$

Next, we expand $d\hat{e}$ in the moving basis

$$d\hat{e} = d\Omega_{ij}\hat{e}_j \tag{2.7}$$

From Eq. (2.6) we see that $d\Omega_{ij}\hat{e}_i\hat{e}_j = 0$ so $d\Omega_{ij}$ must be antisymmetric with respect to $i \leftrightarrow j$:

$$d\Omega_{ij} = -d\Omega_{ji} \tag{2.8}$$

which means that 3×3 matrix $d\Omega_{ij}$ has 3 independent components which can be associated with components of (pseudo) vector $d\vec{\Omega}$

$$d\Omega_1 \equiv d\Omega_{23}, \qquad d\Omega_2 \equiv d\Omega_{31}, \qquad d\Omega_3 \equiv d\Omega_{12} \tag{2.9}$$

With this definition the formula (2.7) can be rewritten as

$$d\hat{e} = d\vec{\Omega}\times\hat{e} \tag{2.10}$$

Indeed,

$$d\hat{e}_1 = d\Omega_{12}\hat{e}_2 + d\Omega_{13}\hat{e}_3 = d\Omega_3\hat{e}_2 - d\Omega_2\hat{e}_3 = \left(d\Omega_1\hat{e}_1 + d\Omega_2\hat{e}_2 + d\Omega_3\hat{e}_3\right)\times\hat{e}_1$$

and similarly for other components.

Geometrical interpretation of vector $d\vec{\Omega}$: $d\vec{\Omega}\times\vec{r}$ describes the following rotation of vector $\vec{r}$: first, on the (infinitesimal) angle $d\Omega_1$ around axis e_1, then on angle $d\Omega_2$ around axis e_2 and finally on $d\Omega_3$ around the axis e_3.

Proof: after the first rotation $\vec{r}\to\vec{r}'$ where

$$\begin{aligned} r_1' &= r_1 \\ r_2' &= r_2\cos d\Omega_1 - r_3\sin d\Omega_1 = r_2 - r_3 d\Omega_1 + O(d\Omega_1^2) \\ r_3' &= r_3\cos d\Omega_1 + r_2\sin d\Omega_1 = r_3 + r_2 d\Omega_1 + O(d\Omega_1^2) \end{aligned} \tag{2.11}$$

After the second rotation $\vec{r}'\to\vec{r}''$ where

$$\begin{aligned} r_1'' &= r_1'\cos d\Omega_2 + r_3'\sin d\Omega_2 \simeq r_1\cos d\Omega_2 + r_3\sin d\Omega_2 \simeq r_1 + r_3 d\Omega_2 \\ r_2'' &= r_2' \simeq r_2 - r_3 d\Omega_1 \\ r_3'' &= r_3'\cos d\Omega_2 - r_1'\sin d\Omega_2 \simeq r_3 + r_2 d\Omega_1 - r_1' d\Omega_2 \end{aligned} \tag{2.12}$$

Finally, after third rotation $\vec{r}'' \to \vec{r}'''$ such that

$$
\begin{aligned}
r_1''' &= r_1'' \cos d\Omega_3 - r_2'' \sin d\Omega_3 \simeq r_1 + r_3 d\Omega_2 - r_2 d\Omega_3 = r_1 + (d\vec{\Omega} \times \vec{r})_1 \\
r_2''' &= r_2'' \cos d\Omega_3 + r_1'' \sin d\Omega_3 \simeq r_2 - r_3 d\Omega_1 + r_1 d\Omega_3 = r_2 + (d\vec{\Omega} \times \vec{r})_2 \\
r_3''' &= r_3'' = r_3 + r_2 d\Omega_1 - r_1 d\Omega_2 = r_3 + (d\vec{\Omega} \times \vec{r})_3
\end{aligned}
\tag{2.13}
$$

Thus, as a result of these three successive rotations, we get the rotation $\vec{r} \to \vec{r}''' = \vec{r} + d\vec{\Omega} \times \vec{r}$. As seen from the definition of the cross product, this is the rotation around the axis defined by $d\hat{\vec{\Omega}}$ on the angle $|d\vec{\Omega}|$:

$$
d\vec{r} = d\vec{\Omega} \times \vec{r} \tag{2.14}
$$

Note that infinitesimal rotations around x, y and z axis commute with one another. This is a general property: given two successive infinitesimal rotations described by $d\vec{\Omega}_1$ and $d\vec{\Omega}_2$, the resulting rotation is

$$
\vec{r}'' \simeq \vec{r}' + d\vec{\Omega}_2 \times \vec{r}' = (\vec{r} + d\vec{\Omega}_1 \times \vec{r}) + d\vec{\Omega}_2 \times (\vec{r} + d\vec{\Omega}_1 \times \vec{r}) \simeq \vec{r} + (d\vec{\Omega}_1 + d\vec{\Omega}_2) \times \vec{r} \tag{2.15}
$$

It should be mentioned, however, that finite rotations do not commute.

Returning to Eq. (2.10) we get

$$
\frac{d\hat{e}}{dt} = \frac{d\vec{\Omega}}{dt} \times \hat{e} = \vec{\omega} \times \hat{e} \tag{2.16}
$$

where

$$
\vec{\omega}(t) = \frac{d\vec{\Omega}}{dt} \tag{2.17}
$$

is the instantaneous angular velocity of the rotating frame as seen from the inertial frame.

Substituting Eq. (2.16) into Eq. (2.5) we get

$$
\left(\frac{d\vec{v}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{v}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{v} \tag{2.18}
$$

Note that we did not use the specific form of $\vec{v}$ in the derivation of Eq. (2.18) which means that it holds true for any vector A measured in there two frames

$$
\left(\frac{d\vec{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{A} \tag{2.19}
$$

In particular, this equation can be applied to $\vec{A} = \vec{\omega}$ and then we get

$$
\left(\frac{d\vec{\omega}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{\omega}}{dt}\right)_{\text{body}} \tag{2.20}
$$

2.2.1 Accelerations

Differentiating formula

$$
\left(\frac{d\vec{r}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{r}}{dt}\right)_{\text{body}} + \omega \times \vec{r} \tag{2.21}
$$

with respect to t, we get

$$
\frac{d^2}{dt^2} r_i(t)\hat{e}_i(t) = \hat{e}_i(t)\frac{d^2}{dt^2} r_i(t) + 2\left(\frac{d}{dt} r_i(t)\right)\frac{d}{dt}\hat{e}_i(t) + r_i(t)\frac{d^2}{dt^2}\hat{e}_i(t) \tag{2.22}
$$

$$\begin{aligned}
\frac{d^2}{dt^2}r_i(t)\hat{e}_i(t) &= \hat{e}_i(t)\frac{d^2}{dt^2}r_i(t) + 2\Big(\frac{d}{dt}r_i(t)\Big)\frac{d}{dt}\hat{e}_i(t) + r_i(t)\frac{d^2}{dt^2}\hat{e}_i(t) \qquad (2.23)\\
&= \hat{e}_i(t)\ddot{r}_i(t) + 2\dot{r}_i(t)(\omega(t)\times\hat{e}_i(t)) + r_i(t)\frac{d}{dt}(\vec{\omega}(t)\times\hat{e}_i(t))\\
{}_i &= \hat{e}_i(t)\ddot{r}_i(t) + 2\dot{r}_i(t)(\omega(t)\times\hat{e}_i(t)) + r_i(t)\big(\frac{d}{dt}\vec{\omega}(t)\big)\times\hat{e}_i(t)) + r_i(t)(\vec{\omega}(t)\times\frac{d}{dt}\hat{e}_i(t))\\
&= \hat{e}_i(t)\ddot{r}_i(t) + 2\dot{r}_i(t)(\omega(t)\times\hat{e}_i(t)) + r_i(t)\big(\dot{\vec{\omega}}(t)\big)\times\hat{e}_i(t)) + r_i(t)(\vec{\omega}(t)\times(\vec{\omega}(t)\times\hat{e}_i(t)))
\end{aligned}$$

or

$$\Big(\frac{d^2}{dt^2}\vec{r}(t)\Big)_{\text{inertial}} = \Big(\frac{d^2}{dt^2}\vec{r}(t)\Big)_{\text{body}} + 2\vec{\omega}\times\big(\frac{d\vec{r}(t)}{dt}\big)_{\text{body}} + \dot{\vec{\omega}}(t)\times\vec{r}(t) + \vec{\omega}(t)\times\big(\vec{\omega}(t)\times\vec{r}(t)\big) \qquad (2.24)$$

Part VI

2.2.2 Translations and rotations

If the origin of the body-fixed frame moves as $\vec{\alpha}(t)$ the relation between $\vec{r}$ and $\vec{r}_0$ is

$$\begin{aligned}
\vec{r}_0 &= \vec{r}(t) + \vec{\alpha}(t) \quad\Rightarrow\quad \vec{r}_0 - \vec{\alpha}(t) = \vec{r} = \hat{e}_i(t)r_i(t)\\
\Leftrightarrow\ r_{0i}^{(0)}(t)\hat{e}_i^{(0)} - \alpha_i^{(0)}(t)\hat{e}_i^{(0)} &= \hat{e}_i(t)r_i(t) \qquad (2.25)
\end{aligned}$$

The change of unit vectors $\hat{e}_i$ is determined only by rotation of the moving frame and does not depend on the translational motion of that frame ⇒ our proof of $\dot{\hat{e}}_i = \vec{\omega}\times\hat{e}_i$ (see Eqs. (2.10-2.16) stays valid in the case of moving origin of the body-fixed frame.

Now, repeating the derivation of Eq. (2.23) we see that

$$\begin{aligned}
\ddot{r}_{0i}^{(0)}(t)\hat{e}_i^{(0)} - \ddot{\alpha}_i^{(0)}(t)\hat{e}_i^{(0)} &= \frac{d^2}{dt^2}r_i(t)\hat{e}_i(t) = \hat{e}_i(t)\ddot{r}_i(t) + 2\dot{r}_i(t)\dot{\hat{e}}_i(t) + r_i(t)\ddot{\hat{e}}_i(t) \qquad (2.26)\\
&= \hat{e}_i(t)\ddot{r}_i(t) + 2\dot{r}_i(t)(\omega(t)\times\hat{e}(t))_i + r_i(t)\big(\dot{\vec{\omega}}(t)\big)\times\hat{e}(t))_i + r_i(t)(\vec{\omega}(t)\times(\vec{\omega}(t)\times\hat{e}(t)))_i
\end{aligned}$$

or

$$\big(\ddot{\vec{r}}_0\big)_{\text{inertial}} = \big(\ddot{\vec{\alpha}}\big)_{\text{inertial}} + \big(\ddot{\vec{r}}\big)_{\text{body}} + 2\big(\omega\times\dot{\vec{r}}\big)_{\text{body}} + \big(\dot{\vec{\omega}}\times\vec{r}\big)_{\text{body}} + \big(\vec{\omega}\times(\vec{\omega}\times\vec{r})\big)_{\text{body}} \qquad (2.27)$$

which is correct for the origin undergoing an arbitrary acceleration $\ddot{\vec{\alpha}}(t)$.

2.2.3 Newton's laws in accelerated coordinate systems

From the 2nd law in the inertial system

$$m\big(\ddot{\vec{r}}_0(t)\big)_{\text{inertial}} = \vec{F}(t) \qquad (2.28)$$

and Eq. (2.27) we get Newton's 2nd law for an observer sitting in the translating and rotating frame

$$m\big(\ddot{\vec{r}}\big)_{\text{body}} = \vec{F} - m\big(\ddot{\vec{\alpha}}\big)_{\text{inertial}} - \underbrace{2m\omega\times\big(\dot{\vec{r}}\big)_{\text{body}}}_{\text{Coriolis force}} - m\dot{\vec{\omega}}\times\vec{r} - \underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{centrifugal force}} \qquad (2.29)$$

(recall that $\big(\dot{\vec{\omega}}\big)_{\text{body}} = \big(\dot{\vec{\omega}}\big)_{\text{inertial}}$).

2.3 Motion on the surface of the Earth

Assume circular orbit of the earth around the sun with radius $R_{\rm se} = 1.5\times10^{11}$m with period $\tau_{\rm se} = 3.16\times10^7$s $\Leftrightarrow$ $\omega_{\rm se} = 2\times10^{-7}{\rm s}^{-1}$, and daily rotation of the earth with $R_{\rm e} = 6.4\times10^6$m, period $\tau_{\rm e} = 8.64\times10^6$s and angular frequency

$$\omega_{\rm e} \;=\; \frac{2\pi}{\tau_{\rm e}} \;=\; 7.3\times10^{-5}\frac{1}{\rm s} \tag{2.30}$$

The inertial frame is fixed at the sun's center and the moving non-inertial frame is fixed in the rotating earth at the origin in the earth's center. The equation (2.29) takes the form

$$m(\ddot{\vec r})_{\rm e} \;=\; \vec F - m(\ddot{\vec\alpha})_{\rm inertial} - 2m(\vec\omega_{\rm e}+\vec\omega_{\rm se})\times(\dot{\vec r})_{\rm e} - m(\vec\omega_{\rm e}+\vec\omega_{\rm se})\times((\vec\omega_{\rm e}+\vec\omega_{\rm se})\times\vec r) \tag{2.31}$$

where $\vec F = \vec F_g^{\rm e} + \vec F^{\rm s} + \vec F'$ is a sum of earth's gravitational force, sun's gravitational force, and any other relevant forces $\vec F'$ (we assume that $\vec\omega_{\rm e}$ and $\vec\omega_{\rm se}$ do not change with time).

Suppose we are considering some object on the earth's surface. Since $\frac{\omega_{\rm se}}{\omega_{\rm e}} \sim 2.73\times10^{-3}$ we can neglect $\vec\omega_{\rm se}$ in the above formula and get

$$m(\ddot{\vec r})_{\rm e} \;=\; \vec F' + \vec F_g^{\rm e} + \vec F^{\rm s} - m(\ddot{\vec\alpha})_{\rm inertial} - 2m\vec\omega_{\rm e}\times(\dot{\vec r})_{\rm e} - m\vec\omega_{\rm e}\times(\vec\omega_{\rm e}\times\vec r) \tag{2.32}$$

For a body on the earth's surface the ratio of the gravitational forces due to the earth and due to the sun is

$$\frac{\vec F_g^{\rm s}}{\vec F_g^{\rm e}} \;=\; \frac{M_{\rm e}R_{\rm s}^2}{M_{\rm s}R_{\rm e}^2} \;\sim\; 1.7\times10^{-3} \tag{2.33}$$

Moreover, at earth center $\vec F^{\rm s}$ would be exactly equal to $m(\ddot{\vec\alpha})_{\rm inertial}$ so we can safely neglect $m(\ddot{\vec\alpha})_{\rm inertial}$ in the above equation and get

$$m(\ddot{\vec r})_{\rm e} \;=\; \vec F' + \vec F_g^{\rm e} - 2m\vec\omega\times(\dot{\vec r})_{\rm e} - m\vec\omega\times(\vec\omega\times\vec r) \tag{2.34}$$

where ω is $\omega_{\rm e}$.

2.3.1 Falling particle

Consider a particle released from height $h \ll R_{\rm e}$ above the earth's surface. From Eq. (2.34) we get

$$m\ddot{\vec r} \;=\; mg - 2m\vec\omega\times\dot{\vec r} \tag{2.35}$$

where

$$\vec g \;=\; -GM_{\rm e}\frac{\vec r}{r^3} - \vec\omega\times(\vec\omega\times\vec r) \tag{2.36}$$

The second term here is the acceleration due to centrifugal force. [3]

We choose a local frame on the earth's surface with $\hat e_x$ southward, $\hat e_y$ eastward, and $\hat e_z$ vertically upward as shown in Fig. 31. We will solve Eq. (2.35) perturbatively, keeping

[3] At the poles, $\vec g$ is radial with the magnitude $GM_{\rm e}\frac{\vec r}{R_e^2}$ but at the equator $\vec g = \left(-GM_{\rm e}\frac{\vec r}{R_e^2} + \omega^2 Re\right)\hat r$. Numerical estimates give the second term of the scale about 3.5 % of the first term.

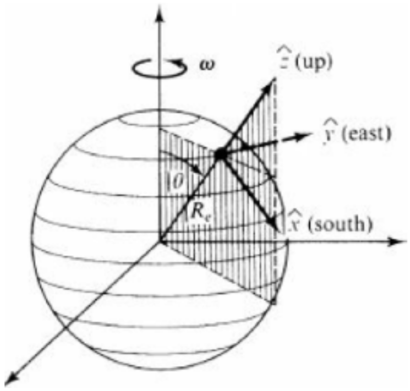

Figure 31. Earth-fixed frame

first two terms of the expansion in ω [4]:

$$\vec{r}(t) \;=\; \vec{r}_0(t) + \vec{r}_1(t) \tag{2.37}$$

In the leading order the Eq. (2.35) reads

$$\ddot{\vec{r}} \;=\; \vec{g} \;=\; -\,g\hat{e}_z \tag{2.38}$$

The correction to g due to centrifugal force is $\sim \omega^2$. It exceeds our accuracy so we can assume

$$\vec{g} \;=\; -\,g\hat{e}_z, \quad g = \frac{GM_e}{R_e^2} \;\sim\; 9.8\frac{\mathrm{m}}{\mathrm{sec}^2} \tag{2.39}$$

and get

$$\vec{r}_0(t) \;=\; \vec{r}(0) - \frac{1}{2}gt^2\hat{z} \tag{2.40}$$

Substituting Eq. (2.37) into Eq. (2.35) we get in the first order in ω

$$\ddot{\vec{r}}_0 + \ddot{\vec{r}}_1 \;=\; -\,g\hat{z} - 2\vec{\omega}\times\dot{\vec{r}}_0 \quad\Rightarrow\quad \ddot{\vec{r}}_1(t) \;=\; 2\vec{\omega}\times\dot{\vec{r}}_0(t) \;=\; -\,2t\vec{\omega}\times\vec{g} \tag{2.41}$$

Using the initial conditions $\vec{r}_1(0) = 0$ and $\dot{\vec{r}}_1(0) = 0$ we get

$$\vec{r}_1(t) \;=\; -\,\frac{t^3}{3}\vec{\omega}\times\vec{g} \;=\; \frac{1}{3}\omega g t^3 \sin\theta \hat{e}_y \tag{2.42}$$

so the total trajectory becomes

$$\vec{r}(t) \;=\; \big(h - \frac{g}{2}t^2\big)\hat{e}_z \;+\; \frac{1}{3}\omega g t^3 \sin\theta \hat{e}_y \tag{2.43}$$

Properties of Eq. (2.43)

- The vertical motion is independent of ω in the first order.

[4]The corresponding dimensionless parameter is ωt_0 where t_0 is the time of free fall from a height h. For h=100m $\omega t_0 = 3.3\times 10^{-4}$

- The particle is deflected eastward and the effect is the same in northern and southern hemisphere. It is maximal at the equator ($\sin\theta = 1$). Foe example, at h =100m the equatorial deflection is 2.2cm.

- The eastward deflection may seem surprising since Earth itself rotates to the east. However, in the inertial frame $\vec{v}_0$ has abn eastward component which increases as r decreases due to conservation of angular momentum.

2.3.2 Horizontal motion

Consider a particle located at polar angle θ moving with horizontal velocity along the direction making angle ϕ with x axis on Fig. 31. We get

$$\hat{\omega} \;=\; -\,\hat{e}_x \sin\theta + \hat{e}_z \cos\theta, \qquad \vec{v} \;=\; v(\hat{e}_x \cos\phi + \hat{e}_y \sin\phi) \tag{2.44}$$

so the Coriolis force (2.29) takes the form

$$\vec{F}_c \;=\; -\,2m\vec{\omega}\times\vec{v} \;=\; 2m\omega v(\hat{e}_x \cos\theta\sin\phi - \hat{e}_y \cos\theta\cos\phi + \hat{e}_z \sin\theta\sin\phi) \tag{2.45}$$

For example, Coriolis force pulls a north-moving particle ($\phi = \pi$) in the northern hemishpere ($\cos\theta > 0$) to the east and south-moving particle to the west. Conversely, in the southern hemisphere a north-moving particle is pulled to the west and a south-moving particelto the east. That is why hurricanes rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere.

2.4 Foucault pendulum

Consider a pendulum composed of massless rigid rod fixed at some support point on one end and with particle of mass m attached to the other end (see Fig. 32). The equation of

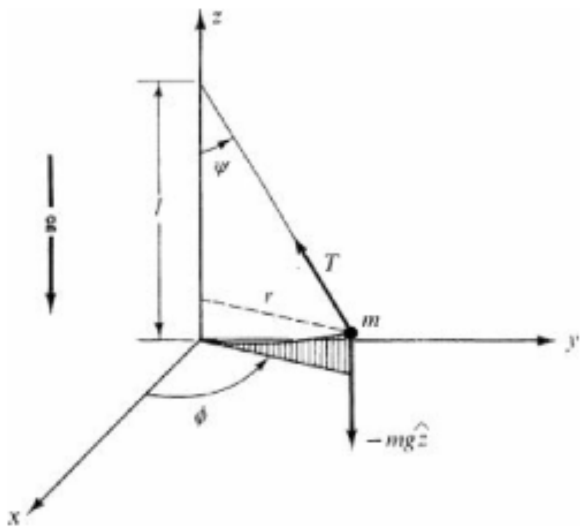

Figure 32. Foucault pendulum

motion is (2.34):

$$m\ddot{\vec{r}} \;=\; \vec{T} + m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}} \tag{2.46}$$

where $\vec{g}$ is given by Eq. (2.36) and the velocity and acceleration are those seen by the terrestrial observer.

Here again we will calculate the motion in the leading order in ω so $\vec{g} = -g\hat{e}_z$. Using the same frame as in Fig. 31 we get

$$\omega\times\dot{\vec{r}} \;=\; -\,\omega\hat{e}_x\cos\theta\dot{y} + \omega(\sin\theta\dot{z} + \cos\theta\dot{x})\hat{e}_y - \omega\sin\theta\dot{y}\hat{e}_z \tag{2.47}$$

and

$$m\vec{g} + \vec{T} \;=\; -\,T\hat{e}_x\sin\psi\cos\phi - T\hat{e}_y\sin\psi\sin\phi + \hat{e}_z(T\cos\psi - mg) \tag{2.48}$$

Consider first the equation for vertical motion

$$m\ddot{z} \;=\; T\cos\psi - mg + 2m\omega\dot{y}\sin\theta \tag{2.49}$$

Since $\omega v \ll 1$ (numerical estimate for $v = 1\frac{\mathrm{m}}{\mathrm{s}}$ is $\omega v \simeq \; 7\times 10^{-5}\frac{\mathrm{m}}{\mathrm{s}^2}$) we can neglect the last term in r.h.s. of Eq. (2.49) in comparison to two other terms. Moreover, the vertical displacement $z = l(1-\cos\psi)$ is small for small displacements form the equilibrium ($r \ll l \Rightarrow \; \psi \simeq \frac{r}{l} \ll 1 \;\Rightarrow\; z \simeq \frac{r^2}{l} \ll l$) so we get approximately

$$T\cos\psi \;\simeq\; T \;\simeq\; mg \tag{2.50}$$

At the next step we consider equations for the horizontal motion

$$m\ddot{x} \;=\; -\,T\sin\psi\cos\phi \;+\; 2m\omega\dot{y}\cos\theta \tag{2.51}$$

$$m\ddot{y} \;=\; -\,T\sin\psi\sin\phi \;-\; 2m\omega(\dot{x}\cos\theta + \dot{z}\sin\theta) \tag{2.52}$$

Note that in the last term in the r.h.s. of Eq. (2.52) the $\dot{z}$ term can be omitted in comparison to $\dot{x}$ term since $z \sim \frac{r^2}{l} \ll x \sim r$. Moreover, since $\sin\psi\cos\phi \simeq \frac{x}{l}$ and $\sin\psi\sin\phi \simeq \frac{y}{l}$ (see Fig. 32) we get from Eqs. (2.50-2.52)

$$\begin{aligned}\ddot{x} \;&=\; -\,\frac{g}{l}x + 2\omega\dot{y}\cos\theta\\ \ddot{y} \;&=\; -\,\frac{g}{l}y - 2\omega\dot{x}\cos\theta\end{aligned} \tag{2.53}$$

where $\omega\cos\theta \;=\; \omega\cdot\hat{e}_z = \omega_\perp$ is the vertical projection of earth's angular velocity.

Trick to solve Eq. (2.53): multiply the second equation by "i" (imaginary unit) and add to the first equation. We get

$$\ddot{\zeta} \;=\; -\,\frac{g}{l}\zeta - 2i\omega\dot{\zeta}\cos\theta \tag{2.54}$$

where

$$\zeta \;\equiv\; x + iy \tag{2.55}$$

The equation (2.54) is a differential equation with constant coefficiens $\Rightarrow$ in can be solved by exponential ansatz $\zeta(t) \;=\; \zeta_0 e^{-i\sigma t}$. We get

$$\sigma^2 - 2\omega\sigma\cos\theta - \frac{g}{l} \;=\; 0 \quad\Rightarrow\quad \sigma_\pm \;=\; \omega_\perp \pm \sqrt{\omega_\perp^2 + \frac{g}{l}} \tag{2.56}$$

so the general solution of Eq. (2.54) can be written as

$$\zeta(t) \;=\; Ae^{-i\omega_\perp t - iqt} + Be^{-i\omega_\perp t + iqt} \qquad q \;\equiv\; \sqrt{\omega_\perp^2 + \frac{g}{l}} \tag{2.57}$$

W.l.o.g. let us assume that the pendulum rod is displaced as small distance a southward and released, then $\zeta(0) = x_0 = a$ and $\dot{\zeta}(0) = 0$ so we get

$$\zeta(t) \;=\; ae^{-i\omega_\perp t}\big(\cos qt + i\frac{\omega_\perp}{q}\sin qt\big) \tag{2.58}$$

Typically, $\omega_\perp^2 \ll \frac{g}{l}$ and the motion is approximately

$$\zeta(t) \;=\; a(\cos qt)e^{-i\omega_\perp t} \tag{2.59}$$

so

$$\begin{aligned} x(t) \;&=\; \Re\zeta(t) \;=\; a(\cos\omega_\perp t)\cos\big(t\sqrt{\frac{g}{l}}\big) \\ y(t) \;&=\; \Im\zeta(t) \;=\; -\,a(\sin\omega_\perp t)\cos\big(t\sqrt{\frac{g}{l}}\big) \end{aligned} \tag{2.60}$$

Since $\omega_\perp \ll \sqrt{\frac{g}{l}}$ these equation represent a superposition of two perpendicular oscillatory motions proportional to $\cos\big(t\sqrt{\frac{g}{l}}\big)$ but with slowly varying amplitudes: $a\cos\omega_\perp t$ and $-a\sin\omega_\perp t$.

Dividing $y(t)$ by $x(t)$ we get

$$\tan\phi \;=\; \frac{y(t)}{x(t)} \;=\; -\tan\omega_\perp t \tag{2.61}$$

Thus, the motion occurs in a plane

$$\phi \;=\; -\,\omega_\perp t \tag{2.62}$$

rotating with angular velocity $\omega_\perp$. This rotation is clockwise (as viewed from above) in the nothern hemisphere and counterclockwise in the southern hemisphere. On the poles, Foucault pendulum would make a full 360° turn exactly in one day (and on the equator it does not rotate).

2.5 Tides

There is another (and very visible) effect due to non-inertial nature of the frame on Earth's surface. From our beach experience we know that every day there are two low tides and two high tides in the ocean. This phenomenon can be explained by analysis of Newton's 2nd law in the Earth frame.

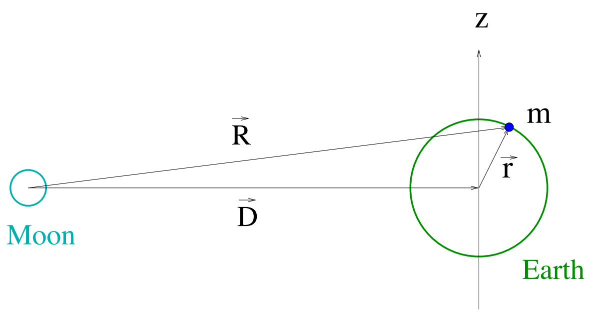

Figure 33. Earth-Moon system

Let us first ignore rotation of Earth about its axis and influence of the Sun. Even in this approximation, Earth is not an inertial frame. The origin of the reference frame attached to the center of Earth is accelerating. This acceleration is due to the gravitational

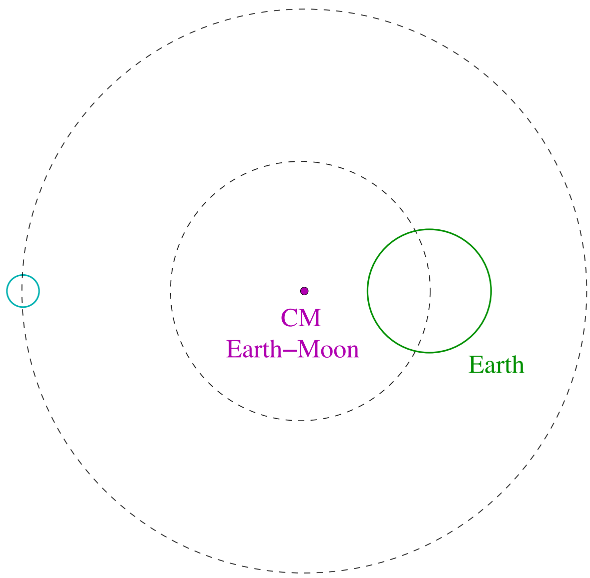

Figure 34. Top view of xy plane of Earth's orbit around CM

force between the Earth and the Moon

$$M_e\vec{a}_e = -GM_eM_m\frac{\vec{D}}{D^3} = -GM_eM_m\frac{\hat{D}}{D^2} \Rightarrow \vec{a}_e = -GM_m\frac{\hat{D}}{D^2} \qquad (2.63)$$

Since we neglect Earth's rotation, Newton's 2nd law (2.29) reduces to

$$m\vec{a} \;=\; \vec{F} - m\vec{a}_{\rm e} \equiv \; \vec{F}_{\rm t} + m\vec{g} \tag{2.64}$$

where $\vec{F}_{\rm t}$ is an attraction force between the body and the Moon minus the $m\vec{a}_{\rm e}$ term. Let us consider the tidal force $\vec{F}_{\rm t}$

$$\vec{F}_{\rm t} \;=\; -GmM_{\rm m}\frac{\hat{R}}{R^2} + GmM_{\rm m}\frac{\hat{D}}{D^2} \tag{2.65}$$

Note that $\vec{F}_{\rm t}$ vanishes at the center of Earth.

It is useful to decompose $\vec{F}_{\rm t}$ into z and y components. Since $\vec{R} = \vec{D} + \vec{r}$ and $\frac{r}{D} \ll 1$

$$\begin{aligned} F^{\rm t}_x \;&=\; -GM_{\rm m}m\Big\{\frac{D+x}{\big((D+x)^2+z^2\big)^{\frac{3}{2}}} - \frac{1}{D^2}\Big\} \\ &=\; -\frac{GM_{\rm m}m}{D^2}\Big\{\frac{1+t_x}{\big((1+t_x)^2+t_z^2\big)^{\frac{3}{2}}} - 1\Big\} \end{aligned} \tag{2.66}$$

where $t_x \equiv \frac{x}{D}$ and $t_z \equiv \frac{z}{D}$. Since $t_x, t_z \ll 1$ we get

$$F^{\rm t}_x \;\simeq\; -\frac{GM_{\rm m}m}{D^2}\big\{(1+t_x)(1-3t_x) - 1\big\} \;\simeq\; \frac{2GM_{\rm m}m}{D^2}t_x \;=\; \frac{2GM_{\rm m}mx}{D^3} \tag{2.67}$$

Similarly,

$$F^{\rm t}_z \;=\; -GM_{\rm m}m\frac{z}{\big((D+x)^2+z^2\big)^{\frac{3}{2}}} \;=\; -\frac{GM_{\rm m}m}{D^2}\frac{t_z}{\big((1+t_x)^2+t_z^2\big)^{\frac{3}{2}}} \;=\; -\frac{GM_{\rm m}mz}{D^3} \tag{2.68}$$

The cartoon of these tidal forces is shown in Fig. 35.

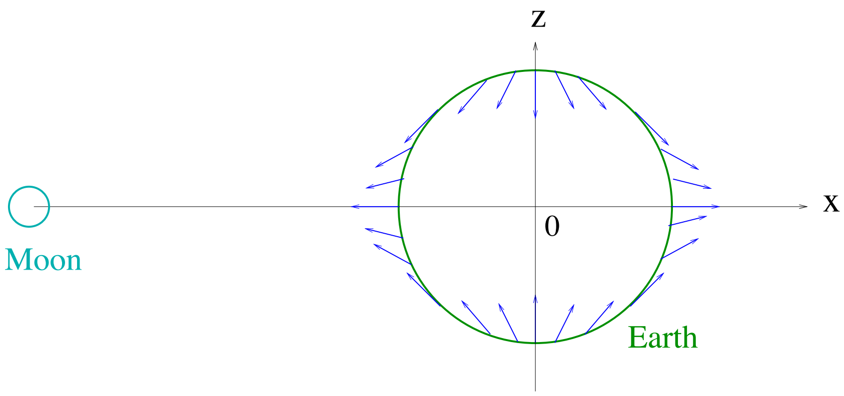

Figure 35. Two tides

If the earth was rigid, the tidal forces would have no effect on it, but the water in the oceans is free to move around, so it bulges around the Earth-Moon line. As the Earth rotates, the person on the surface sees the bulge rotating in opposite direction, so at a given spot one sees two low tides and two high tides per day [5].

In addition, there are tides due to Sun's influence. One may expect them to be even bigger than the tidal waves due to the Moon influence since the ratio of forces of gravity is

$$\frac{F_{\rm Sun}}{F_{\rm Moon}} \;=\; \Big(\frac{GM_{\rm s}}{R^2_{\rm es}}\Big) : \Big(\frac{GM_{\rm m}}{R^2_{\rm ms}}\Big) \;\simeq\; 175 \tag{2.69}$$

However, the explicit formulas for the tidal forces (2.68) and (2.69) tell us that

$$\frac{F_{\rm Sun}}{F_{\rm Moon}} \;=\; \Big(\frac{GM_{\rm s}}{R^3_{\rm es}}\Big) : \Big(\frac{GM_{\rm m}}{R^3_{\rm ms}}\Big) \;\simeq\; 0.45 \tag{2.70}$$

so the tide due to the Moon is twice as big as the tide due to the Sun. These two tides may add up if the Moon is close to the Sun-Earth line, or partially cancel if the Moon is at 90° with respect to Earth-Sun vector.

Part VII

3 Lagrangian dynamics

3.1 Generalized coordinates

Consider a particle moving in 3 dimensions under the action of a force $\vec{F}$:

$$m\ddot{\vec{r}} \;=\; \vec{F} \tag{3.1}$$

If the position of the particle is completely specified by the three components $\vec{r}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y + z(t)\hat{e}_z$ we say that the particle has three degrees of freedom. Given $\vec{r}(0) = \vec{r}_0$ and $\dot{\vec{r}}(0) = \dot{\vec{r}}_0 \equiv \vec{v}_0$ we can predict its motion at all later times because the solution of the second-order differential equation (3.1) is uniquely defined by initial conditions $\vec{r}_0$ and $\vec{v}_0$.

However, if we consider a particle being constrained to slide along a wire, it is sufficient to specify the position of the particle on the wire so we have a system with one degree of freedom.

[5] Strictly speaking, the moon rotates about the Earth so in 28 days we will see only 55 pairs of tides.

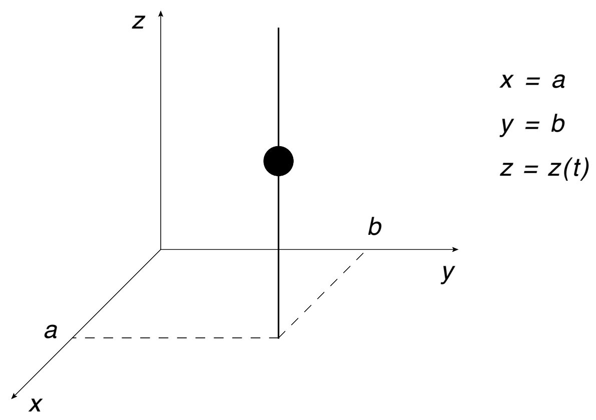

Figure 36. Constrained motion: example 1

The position of the particle is=n 3-dim space is given by three coordinates x, y, and z but there are two constraints

- #1: $x = a$
- # 2: $y = b$

so the position of the particle is specified by a single "generalized coordinate" $z = z(t)$.

The constraints may change in time, for example:

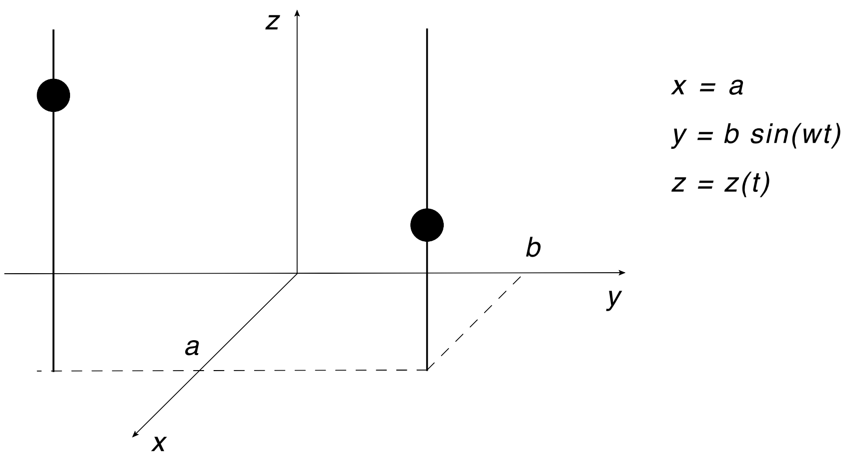

Figure 37. Constrained motion; example 2

These ideas can be generalized for a system of N particles. A "configuration" of the system is specified by $3N$ Cartesian coordinates. However, they may be not all independent due to the presence of some constraints. These constraints may be specified by equations of the type

$$\left.\begin{array}{rcl} f_1(x_1, x_2, ...x_{3N}, t) & = & 0 \\ f_2(x_1, x_2, ...x_{3N}, t) & = & 0 \\ & . & \\ & . & \\ & . & \\ f_k(x_1, x_2, ...x_{3N}, t) & = & 0 \end{array}\right\} \quad \text{k constraints, } k \leq 3N \tag{3.2}$$

Here we used the notation $x_1 \equiv x_1, x_2 \equiv y_1, x_3 \equiv z_1$ for particle # 1, $x_4 \equiv x_2, x_5 \equiv y_2, x_3 \equiv z_2$ for particle # 2, ..., and $x_{3N-2} \equiv x_N, x_{3N-1} \equiv y_N, x_{3n} \equiv z_N$ for particle # N. of some constraints. These constraints may be specified by equations of the type

$$\begin{array}{rl} x_1 \equiv x_1,\ x_2 \equiv y_1,\ x_3 \equiv z_1 & \text{for particle \#1} \\ x_4 \equiv x_2,\ x_5 \equiv y_2,\ x_6 \equiv z_2 & \text{for particle \#2} \\ & . \\ & . \\ x_{3N-2} \equiv x_N,\ x_{3N-1} \equiv y_N,\ x_{3n} \equiv z_N & \text{for particle \#N} \end{array} \tag{3.3}$$

Because of the k constraints, there are $3N - k$ independent coordinates. These are the "generalized coordinates" $q_1, q_2, ...q_{3N-k}$ and the system has $3N - k$ degrees of freedom.

For example, consider the system of the wedge and the block that slides along the incline of the wedge. This system has two degrees of freedom, and generalized coordinates

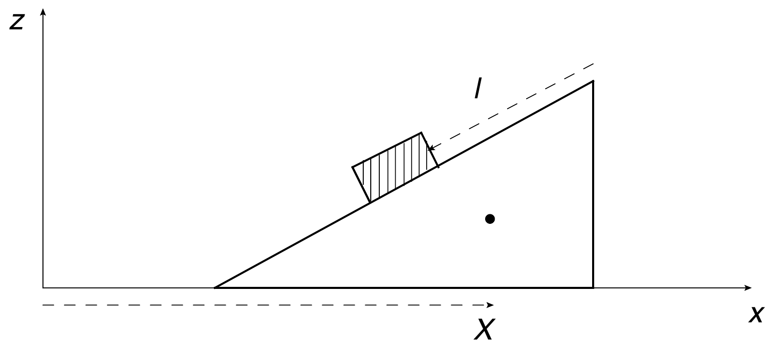

Figure 38. Constrained motion; example 3

can be chosen as the position of the center of mass of the wedge X and the position of the block along the incline l.

NB: Not all constraints can be expressed in the form (3.2). If it is possible to do so, the constraints are called *holonomic constraints*. In some instances, this is not the case.

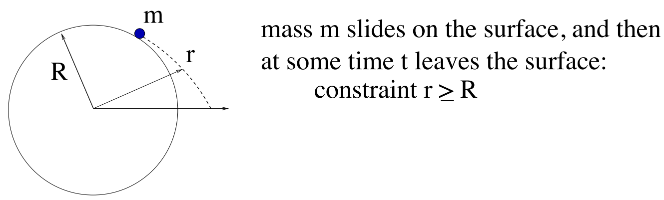

Figure 39. Non-holonomic constraint $r \geq R$

For example, the point mass m sliding on the surface under the weight of gravity has a non-holonomic constraint $r \geq R$.

Another example: wheel rolling on the surface without skidding.

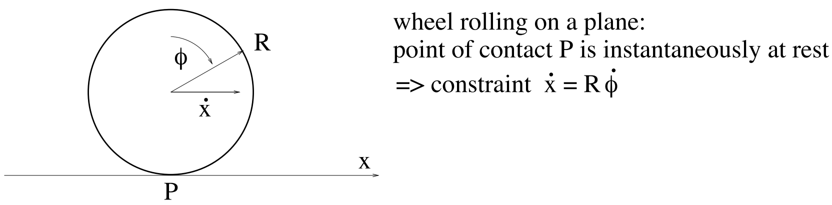

Figure 40. Non-holonomic constraint $\dot{x} = R\dot{\phi}$

NB: The constraints exert forces which are not known *a priori*.

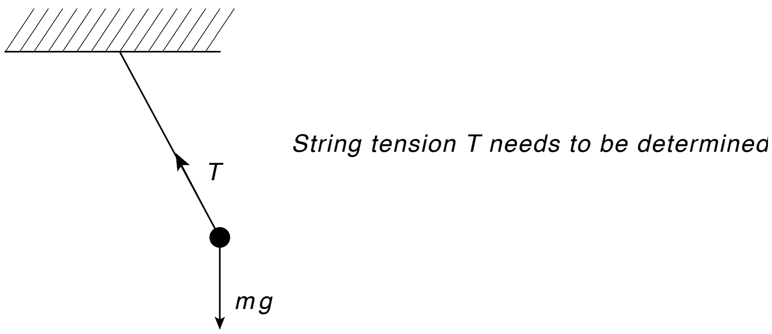

Figure 41. Constrained motion: pendulum

We want to formulate the dynamics in such a way that these forces do not appear in the equations of motion explicitly.

To carry out this program, first we define the virtual displacement δx: δx_i is an infinitesimal displacement of the coordinates consistent with the constraints.

NB: If constraints depend on time t the virtual displacement is taken at a fixed time t so the time is "frozen".

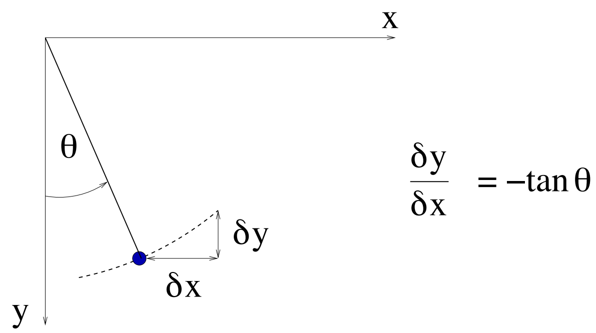

Figure 42. Example: displacements with the constraint $\sqrt{x^2+y^2} = \text{const}$

In terms of the generalized coordinates

$$x_i = x_i(q_1, q_2, ...q_{3N-k}; t), \qquad i = 1, 2, ...N \tag{3.4}$$

⇒

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \frac{\partial x_i}{\partial q_2}\delta q_2 + ... + \frac{\partial x_i}{\partial q_{3N-k}}\delta q_{3N-k} = \sum_{n=1}^{3N-k} \frac{\partial x_i}{\partial q_n}\delta q_n \tag{3.5}$$

Note that change of the coordinates $dx_i = x_i(t+dt) - x_i(t)$ is

$$dx_i = \sum_{n=1}^{3N-k} \frac{\partial x_i}{\partial q_n}\delta q_n + \frac{\partial x_i}{\partial t} = \delta x_i + \frac{\partial x_i}{\partial t}dt \tag{3.6}$$

d'Alembert principle: the forces due to the constraints do no work (friction is ignored here). For example, in the case of pendulum, string tension T is orthogonal to the displacement ⇒ does no work:

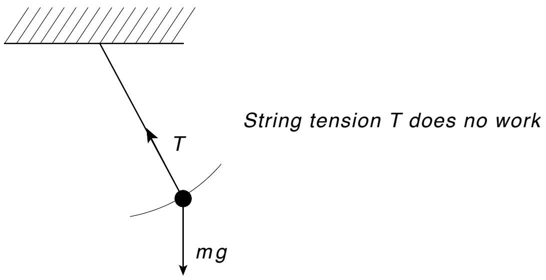

Figure 43. Constrained motion: pendulum

D'Alembert principle can be now used to eliminate the forces of constraints from the dynamical equations. Consider a set of Newton's 2nd laws:

$$\dot{\vec{p}}_j = \vec{F}_j + \vec{R}_j, \qquad j = 1, 2, ..., N \tag{3.7}$$

where $\vec{R}_j$ are forces due to constraints and $\vec{F}_j$ are other (known) forces. In our notations (3.3) the above equation reads

$$\dot{p}_i = F_i + R_i, \qquad i = 1, 2, ..., 3N \tag{3.8}$$

and therefore

$$\sum_{i=1}^{3N}(F_i + R_i - \dot{p}_i)\delta x_i \;=\; 0 \tag{3.9}$$

Note that

$$\sum_{i=1}^{3N} R_i\delta x_i \;=\; \sum_{j=1}^{N} \vec{R}_j \cdot \delta\vec{x}_j \;=\; 0 \tag{3.10}$$

due to d'Alembert principle (forces of constraints do no work).

Thus, the equation (3.9) reduces to

$$\sum_{i=1}^{3N}(F_i - \dot{p}_i)\delta x_i \;=\; 0 \tag{3.11}$$

where the forces due to constraints has been removed. Note, however, that in the presence of constraints the displacements δx_i $(i = 1, 2, ..., 3N)$ are not independent, for example $\delta y = -\delta x \tan\theta$ in Fig. 42.

3.2 Euler-Lagrange equations

Let us rewrite Eq. (3.11) as

$$\sum_{i=1}^{3N} \dot{p}_i\delta x_i \;=\; \sum_{i=1}^{3N} F_i\delta x_i \tag{3.12}$$

and consider each term in turn. However, first we need to find the relation between partial derivatives of usual and generalized coordinates and velocities

From Eq. (3.6) we get

$$\dot{x}_i \;=\; \sum_{n=1}^{3N-k} \frac{\partial x_i}{\partial q_n}\dot{q}_n + \frac{\partial x_i}{\partial t} \tag{3.13}$$

Note that due to Eq. (3.2) $\frac{\partial x_k}{\partial q_k}$ is a function of the generalized coordinates and time $\frac{\partial x_k}{\partial q_k} = f(q_1, ...q_{3N-k}; t)$. If we consider q_k and $\dot{q}_k$ to be independent variables, the partial derivative of the l.h.s. of Eq. (3.13) with respect to $\dot{q}_k$ is simply

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} \;=\; \frac{\partial x_i}{\partial q_j} \tag{3.14}$$

because $\frac{\partial x_i}{\partial q_j}$ in the r.h.s. of Eq. (3.13) does not have an explicit dependence on $\dot{q}_k$.

Next, we consider the lh.s. and the r.h.s. of Eq. (3.9) in turn.

3.2.1 LHS of Eq. (3.12)

$$\sum_{i=1}^{3N} \dot{p}_i\delta x_i \;=\; \sum_{i=1}^{3N} m_i\ddot{x}_i\delta x_i \;=\; \sum_{i=1}^{3N} m_i \sum_{n=1}^{3N-k} \frac{d\dot{x}_i}{dt}\frac{\partial x_i}{\partial q_n}\delta q_n \tag{3.15}$$

where we used Eq. (3.5). Next, we rewrite the r.h.s of this equation as follows

$$\delta x_i \;=\; \sum_{i=1}^{3N} m_i \sum_{n=1}^{3N-k} \left[\frac{d}{dt}\left(\dot{x}_i\frac{\partial x_i}{\partial q_n}\right) - \dot{x}_i\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_n}\right)\right]\delta q_n \tag{3.16}$$

and use Eq. (3.14):

$$\begin{aligned}\delta x_i &= \sum_{i=1}^{3N} m_i \sum_{n=1}^{3N-k} \Big[\frac{d}{dt}\Big(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_n}\Big) - \dot{x}_i \frac{d}{dt}\Big(\frac{\partial x_i}{\partial q_n}\Big)\Big]\delta q_n \\ &= \sum_{n=1}^{3N-k}\Big[\sum_{i=1}^{3N} m_i \frac{d}{dt}\frac{\partial}{\partial \dot{q}_n}\Big(\frac{\dot{x}_i^2}{2}\Big) - \sum_{i=1}^{3N} m_i \dot{x}_i \frac{d}{dt}\Big(\frac{\partial x_i}{\partial q_n}\Big)\Big]\delta q_n \\ &= \sum_{n=1}^{3N-k} \frac{\partial}{\partial \dot{q}_n}\frac{d}{dt}\Big(\sum_{i=1}^{3N} m_i \frac{\dot{x}_i^2}{2}\Big) - \sum_{i=1}^{3N}\sum_{n=1}^{3N-k} m_i \dot{x}_i \frac{d}{dt}\Big(\frac{\partial x_i}{\partial q_n}\Big)\Big]\delta q_n \end{aligned} \tag{3.17}$$

Now, $x_i = x_i(q_1, ...q_{3N-k}, t)$

$$\Rightarrow \frac{d}{dt}\frac{\partial x_i(\{q_j\},t)}{\partial q_n} = \sum_{l=1}^{3N-k} \frac{\partial^2 x_i(\{q_j\},t)}{\partial q_l \partial q_n}\dot{q}_l + \frac{\partial^2 x_i}{\partial q_n \partial t} \tag{3.18}$$

On the other hand, from Eq. (3.13)

$$\frac{\partial}{\partial q_n}\dot{x}_i = \sum_{l=1}^{3N-k} \frac{\partial^2 x_i}{\partial q_l \partial q_n}\dot{q}_l + \frac{\partial^2 x_i}{\partial q_n \partial t} \tag{3.19}$$

so we get $\frac{d}{dt}\frac{\partial x_i}{\partial q_n} = \frac{\partial}{\partial q_n}\dot{x}_i$. Using this formula, we can rewrite the second term in Eq. (3.17) as

$$-\sum_{i=1}^{3N}\sum_{n=1}^{3N-k} m_i \dot{x}_i \frac{d}{dt}\Big(\frac{\partial x_i}{\partial q_n}\Big) = -\sum_{n=1}^{3N-k}\sum_{i=1}^{3N} m_i \dot{x}_i \frac{\partial}{\partial q_n}\frac{d}{dt}x_i = -\sum_{n=1}^{3N-k}\frac{\partial}{\partial q_n}\Big(\sum_{i=1}^{3N}\frac{m_i \dot{x}_i^2}{2}\Big) \tag{3.20}$$

and get

$$\begin{aligned}\delta x_i &= \sum_{n=1}^{3N-k} \delta q_n \frac{\partial}{\partial \dot{q}_n}\frac{d}{dt}\Big(\sum_{i=1}^{3N} m_i \frac{\dot{x}_i^2}{2}\Big) - \sum_{n=1}^{3N-k}\delta q_n \frac{\partial}{\partial q_n}\Big(\sum_{i=1}^{3N}\frac{m_i \dot{x}_i^2}{2}\Big) \\ &= \sum_{n=1}^{3N-k}\delta q_n \Big[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} T\big(\{q_j\},\{\dot{q}_j\},t\big) - \frac{\partial}{\partial q_n} T\big(\{q_j\},\{\dot{q}_j\},t\big)\Big]\end{aligned} \tag{3.21}$$

where

$$T\big(\{q_j\},\{\dot{q}_j\},t\big) = \sum_{i=1}^{3N}\frac{m_i \dot{x}_i^2}{2} \tag{3.22}$$

is the kinetic energy of the system considered as a function of independent variables q_i, $\dot{q}_i$, and t.

3.2.2 RHS of Eq. (3.12)

Now we turn our attention to the r.h.s. of Eq. (3.12). From Eq. (3.5) we get

$$\sum_{i=1}^{3N} F_i \delta x_i = \sum_{i=1}^{3N} F_i \sum_{n=1}^{3N-k} \delta q_n \frac{\partial x_i}{\partial q_n} = \sum_{n=1}^{3N-k} \delta q_n \Big(\sum_{i=1}^{3N}\frac{\partial x_i}{\partial q_n}F_i\Big) \tag{3.23}$$

The expressions in the parenthesis are called "generalized forces"

$$Q_n \equiv \sum_{i=1}^{3N} \frac{\partial x_i}{\partial q_n} F_i \tag{3.24}$$

The generalized forces can be calculated directly from this definition. Alternatively, they can be found from the virtual work done by forces $\vec{F}_j$ for virtual displacement along a given generalized coordinate.

A very important special case is the case of conservative forces

$$F_i(x_1, ...x_{3N}) \;=\; -\frac{\partial}{\partial x_i} V(x_1, ...x_{3N}) \tag{3.25}$$

where the potential V depends only on the positions of the particles. In terms of the generalized coordinates

$$V(x_1, ...x_{3N}) \;=\; V(q_1, ...q_{3N-k}; t) \tag{3.26}$$

so the generalized forces (3.23) can be represented as partial derivatives of potential energy with respect to generalized coordinates

$$Q_n \;=\; -\sum_{i=1}^{3N} \frac{\partial x_i}{\partial q_n} \frac{\partial}{\partial x_i} V(x_1, ...x_{3N})\Big|_{x_i = x_i(\{q_j\}, t)} \;=\; -\frac{\partial}{\partial q_n} V(q_1, ...q_{3N-k}; t) \tag{3.27}$$

3.3 Lagrange equations

Now we are in the position to derive Lagrange equations for dynamics in terms of generalized coordinates. Combining Eqs. (3.12), (3.21), and (3.23) we get

$$\sum_{n=1}^{3N-k} \delta q_n \Big[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} T\big(\{q_j\}, \{\dot{q}_j\}, t\big) - \frac{\partial}{\partial q_n} T\big(\{q_j\}, \{\dot{q}_j\}, t\big)\Big] \;=\; \sum_{n=1}^{3N-k} \delta q_n Q_n\big(\{q_j\}, \{\dot{q}_j\}, t\big) \tag{3.28}$$

Since the displacements δq_n are independent we get the Lagrange equation in the form

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} T\big(\{q_j\}, \{\dot{q}_j\}, t\big) - \frac{\partial}{\partial q_n} T\big(\{q_j\}, \{\dot{q}_j\}, t\big) \;=\; Q_n\big(\{q_j\}, \{\dot{q}_j\}, t\big) \tag{3.29}$$

For the special case of conservative forces the Lagrange equations take the form

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} T\big(\{q_j\}, \{\dot{q}_j\}, t\big) - \frac{\partial}{\partial q_n} T\big(\{q_j\}, \{\dot{q}_j\}, t\big) \;=\; -\frac{\partial}{\partial q_n} V(\{q_j\}; t) \tag{3.30}$$

which can be rewritten as of Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_n} L\big(\{q_j\}, \{\dot{q}_j\}, t\big) = \frac{\partial}{\partial q_n} L\big(\{q_j\}, \{\dot{q}_j\}, t\big) \tag{3.31}$$

where the function which can be rewritten as of Euler-Lagrange equations

$$L\big(\{q_j\}, \{\dot{q}_j\}, t\big) \;\equiv\; T\big(\{q_j\}, \{\dot{q}_j\}, t\big) - V\big(\{q_j\}, t\big) \tag{3.32}$$

is called the *Lagrangian*.

Part VIII

3.4 Examples of Lagrangians

3.4.1 Example 1: double pendulum

Consider double pendulum oscillating in XY plane

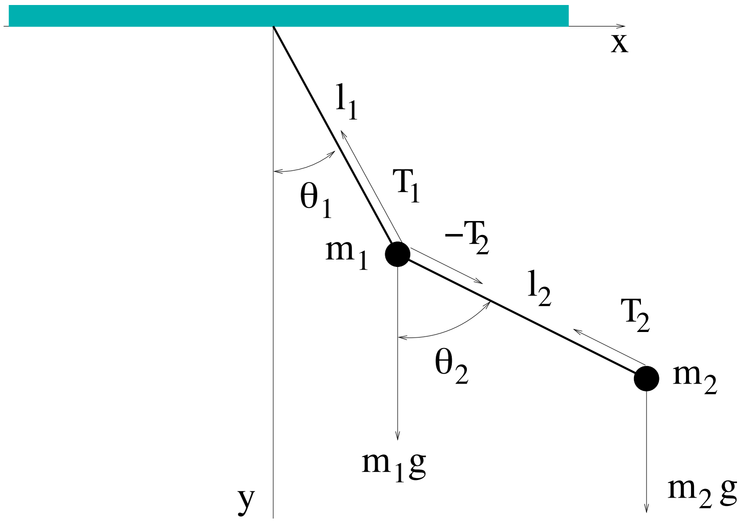

Figure 44. Double pendulum

One can choose the generalized coordinates as θ_1 and θ_2. Apart from the constraint forces $\vec{T}_1$ and $\vec{T}_2$, the only one other is the conservative gravitational force with the potential

$$V \;=\; -m_1 g y_1 - m_2 g y_2 \tag{3.33}$$

In terms of generalized coordinates

$$\begin{aligned} x_1 &= l_1 \sin\theta_1 & y_1 &= l_1 \cos\theta_1 \\ x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 & y_2 &= l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{aligned} \tag{3.34}$$

The Lagranfian is

$$L \;=\; T - V \;=\; \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) + m_1 g y_1 + m_2 g y_2 \tag{3.35}$$

which we need to represent in terms of θ_1, θ_2 and $\dot{\theta}_1$, $\dot{\theta}_2$. From Eq. (3.34) we get

$$\begin{aligned} \dot{x}_1 &= l_1 \dot{\theta}_1 \cos\theta_1 & y_1 &= -\,l_1 \dot{\theta}_1 \sin\theta_1 \\ \dot{x}_2 &= l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2 & \dot{y}_2 &= -\,l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \dot{\theta}_2 \sin\theta_2 \end{aligned} \tag{3.36}$$

and the Lagrangian (3.35) takes the form

$$L \;=\; \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2}\left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right] + \; m g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \tag{3.37}$$

The Euler-Lagrange equation are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial L}{\partial \theta_1}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = \frac{\partial L}{\partial \theta_2}, \tag{3.38}$$

The derivatives are

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - (m_1+m_2) g l_1 \sin\theta_1 \\
\frac{\partial L}{\partial \theta_2} &= m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - m_2 g l_2 \sin\theta_2 \\
\frac{\partial L}{\partial \dot\theta_1} &= (m_1+m_2) l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1-\theta_2) \\
\frac{\partial L}{\partial \dot\theta_2} &= m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1 \cos(\theta_1-\theta_2)
\end{aligned} \tag{3.39}$$

so the Euler-Lagrange equations (3.38) take the form

$$\begin{aligned}
(m_1+m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g l_1 \sin\theta_1 &= 0 \\
m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1-\theta_2) + m_2) g l_2 \sin\theta_2 &= 0
\end{aligned} \tag{3.40}$$

In summary, the equations of motion are

$$\begin{aligned}
\ddot\theta_1 + \frac{m_2 l_2}{(m_1+m_2) l_1}\left[\ddot\theta_2 \cos(\theta_1-\theta_2) + \dot\theta_2^2 \sin(\theta_1-\theta_2)\right] + \frac{g}{l_1}\sin\theta_1 &= 0 \\
\ddot\theta_2 + \frac{l_1}{l_2}\left[\ddot\theta_1 \cos(\theta_1-\theta_2) - \dot\theta_1^2 \sin(\theta_1-\theta_2)\right] + \frac{g}{l_2}\sin\theta_2 &= 0
\end{aligned} \tag{3.41}$$

It is a set of coupled differential equations. Solving them for a given $\theta_{10}, \dot\theta_{10}$ and $\theta_{20}, \dot\theta_{20}$ we can find $\theta_1(t)$ and $\theta_2(t)$.

Let us consider the case of small oscillations $\theta_1, \theta_2 \ll 1$ and $\dot\theta_1, \dot\theta_2 \tau \ll 1$ where τ is the characteristic time for the oscillations (we will see below that $\tau \sim \frac{1}{\omega_0} \sim \sqrt{\frac{l}{g}}$). In this limit the Eqs. (3.41) turn to

$$\begin{aligned}
\ddot\theta_1 + \frac{m_2 l_2}{(m_1+m_2) l_1}\ddot\theta_2 + \frac{g}{l_1}\theta_1 &= 0 \\
\ddot\theta_2 + \frac{l_1}{l_2}\ddot\theta_1 + \frac{g}{l_2}\theta_2 &= 0
\end{aligned} \tag{3.42}$$

For simplicity, let us take $m_1 = m_2 = m$ and $l_1 = l_2 = l$ and define $\omega_0^2 \equiv \frac{g}{l}$. One obtains

$$\begin{aligned}
\ddot\theta_1 + \frac{1}{2}\ddot\theta_2 + \omega_0^2 \theta_1 &= 0 \\
\ddot\theta_2 + \ddot\theta_1 + \omega_0^2 \theta_2 &= 0
\end{aligned} \tag{3.43}$$

Ansatz: $\theta_1 = \rho_1 \cos(\omega t + \phi)$, $\theta_2 = \rho_2 \cos(\omega t + \phi)$

$$\begin{aligned}
\rho_1 \omega^2 + \frac{\rho_2}{2}\omega^2 &= \omega_0^2 \rho_1 \\
\rho_1 \omega^2 + \rho_2 \omega^2 &= \omega_0^2 \rho_2
\end{aligned} \tag{3.44}$$

The solutions do exist only and only if

$$\det \begin{vmatrix} \omega_0^2 - \omega^2 & -\frac{\omega^2}{2} \\ -\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0 \tag{3.45}$$

We will study this case of small oscillations later.

3.4.2 Example 2: pendulum with sliding pivot

Consider a pendulum with pivot at mass m_1 which can slide along the wire in x direction without friction. Again, the motion is supposed to be restricted to XY plane. There are

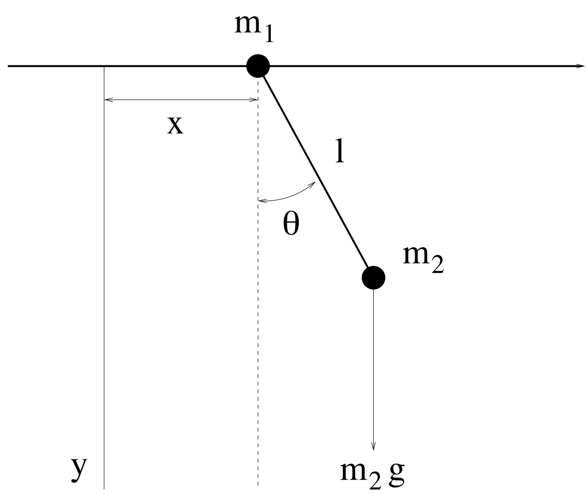

Figure 45. Sliding pendulum

two degrees of freedom which cam be chosen as x and θ:

$$\begin{cases} x_1 = x \\ y_1 = 0 \end{cases} \qquad \begin{cases} x_2 = x + l\sin\theta \\ y_2 = \quad l\cos\theta \end{cases} \tag{3.46}$$

The kinetic energy takes the form

$$T = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}(l^2\dot{\theta}^2 + 2l\cos\theta\dot{x}\dot{\theta}) \tag{3.47}$$

so the Lagrangian in terms of generalized coordinates (3.46) reads

$$L = T - V = L + m_2 g y_2 = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}(l^2\dot{\theta}^2 + 2l\cos\theta\dot{x}\dot{\theta}) + m_2 gl\cos\theta \tag{3.48}$$

The partial derivatives are

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= (m_1 + m_2)\dot{x} + m_2 l\dot{\theta}\cos\theta & \frac{\partial L}{\partial x} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= m_2 l^2\dot{\theta} + m_2 l\dot{x}\cos\theta & \frac{\partial L}{\partial \theta} &= -m_2 l\dot{\theta}\dot{x}\sin\theta - m_2 gl\sin\theta \end{aligned} \tag{3.49}$$

so the Euler-Lagrange equations take the form

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{\partial L}{\partial x} \quad \Rightarrow \quad \frac{d}{dt}\big[(m_1+m_2)\dot{x} + m_2 l\dot{\theta}\cos\theta\big] \;=\; 0 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= \frac{\partial L}{\partial \theta} \quad \Rightarrow \quad \frac{d}{dt}\Big(m_2 l^2\dot{\theta} + m_2 l\dot{x}\cos\theta\Big) \;=\; -\,m_2 l\dot{\theta}\dot{x}\sin\theta - m_2 g l\sin\theta \end{aligned} \tag{3.50}$$

The first of these equations implies that $(m_1+m_2)\dot{x}+m_2l\dot{\theta}\cos\theta$ is a constant. It is easy to see that this constant is equal to the x-component of the total momentum of two particles: $(m_1+m_2)\dot{x}+m_2l\dot{\theta}\cos\theta \;=\; m_1\dot{x}_1+m_2\dot{x}_2 = P_x$. The total momentum of the two particles is conserved since the only *external* forces acting on the pendulum are the force of gravity and normal force at the pivot m_1 and both of them are orthogonal to x axis (recall that we ignore friction at the pivot).

The second equation (3.49) can be simplified to

$$\ddot{\theta} + \frac{\cos\theta}{l}\ddot{x} + \omega_0^2\sin\theta \;=\; 0, \qquad\qquad \omega_0^2 \equiv \frac{g}{l} \tag{3.51}$$

Next, we can express $\ddot{x}$ in terms of θ using first Eq. (3.50)

$$\ddot{x} \;=\; \frac{m_2 l}{m_1+m_2}\dot{\theta}^2\sin\theta - \frac{m_2 l}{m_1+m_2}\ddot{\theta}\cos\theta \tag{3.52}$$

so the equation (3.51) can be rewritten as

$$\Big(1 - \frac{m_2\cos^2\theta}{m_1+m_2}\Big)\ddot{\theta} + \frac{m_2\sin\theta\cos\theta}{m_1+m_2}\dot{\theta}^2 + \omega_0^2\sin\theta \;=\; 0 \tag{3.53}$$

For the case of small oscillations ($\theta \ll 1$)the above equation reduces to

$$\frac{m_1}{m_1+m_2}\ddot{\theta} + \frac{m_2}{m_1+m_2}\theta\dot{\theta}^2 + \omega_0^2\theta \;=\; 0 \tag{3.54}$$

If we assume also that $\dot{\theta} \ll \omega_0$ we get

$$\frac{m_1}{m_1+m_2}\ddot{\theta} + \omega_0^2\theta \;=\; 0 \quad \Leftrightarrow \quad \ddot{\theta} \;=\; -\,\omega_0^2\Big(1+\frac{m_2}{m_1}\Big)\theta \tag{3.55}$$

which is the harmonic equation for oscillations with frequency

$$\omega = \omega_0\sqrt{1+\frac{m_2}{m_1}} \tag{3.56}$$

Check: in the limit $m_1 \to \infty$ we get $\omega \to \omega_0$ for the standard pendulum.

3.5 Calculus of variations

Consider the following mathematical problem: find a function $y(x)$ in the interval $[x_1, x_2]$ such that the integral

$$I \;=\; \int_{x_1}^{x_2} f(y, y', x)dx, \qquad y' \equiv \frac{dy}{dx} \tag{3.57}$$

is at extremum ($\equiv$ minimum or maximum). The integral (3.57) is an example of a *functional* - function of a function. The integral $I\big(y(x)\big)$ is a number which depends on the form of the function $y(x)$.

Let us find the condition for a path $\bar{y}(x)$ to make I stationary ($\equiv$ minimal or maximal). Suppose the function $\bar{y}(x)$ makes I stationary. Let us take an arbitrary function $y(x)$ and let us consider a set of functions $y(x, \alpha)$ such that

$$y(x,\alpha) \;\equiv\; \bar{y}(x) \;+\; \alpha\big(y(x) - \bar{y}(x)\big) \tag{3.58}$$

If $\bar{y}(x)$ makes I stationary, the integral

$$I(\alpha) \;=\; \int_{x_1}^{x_2} f\big[y(x,\alpha), y'(x,\alpha), x\big] dx \tag{3.59}$$

must have an extremum at the point $\alpha = 0$ - otherwise small deviations of the function $\alpha\big(y(x) - \bar{y}(x)\big)$ would lead to a change in the value of $I(\alpha)$. Thus, the necessary condition for $\bar{y}(x)$ to be an extremum of functional (3.57) is

$$\frac{dI(\alpha)}{d\alpha}\bigg|_{\alpha=0} \;=\; 0 \qquad\qquad \text{for any } y(x) \tag{3.60}$$

Taking the derivative of Eq. (3.59) we get

$$\begin{aligned}
\frac{dI(\alpha)}{d\alpha}\bigg|_{\alpha=0} \;&=\; \int_{x_1}^{x_2} \bigg\{ \frac{df(y,y',x)}{dy} \frac{dy(x,\alpha)}{d\alpha}\bigg|_{\alpha=0} + \frac{df(y,y',x)}{dy'} \frac{dy'(x,\alpha)}{d\alpha}\bigg|_{\alpha=0} \bigg\} dx \\
&= \int_{x_1}^{x_2} \bigg\{ \frac{df(y,y',x)}{dy}\bigg|_{y=\bar{y}} \big(y(x) - \bar{y}(x)\big) + \frac{df(y,y',x)}{dy'}\bigg|_{y=\bar{y}} \big(y'(x) - \bar{y}'(x)\big) \bigg\} dx \\
&= \int_{x_1}^{x_2} \bigg\{ \frac{df(y,y',x)}{dy}\bigg|_{y=\bar{y}} \big(y(x) - \bar{y}(x)\big) + \frac{df(y,y',x)}{dy'}\bigg|_{y=\bar{y}} \frac{d}{dx}\big(y(x) - \bar{y}(x)\big) \bigg\} dx
\end{aligned} \tag{3.61}$$

Next, we use integration by parts in the second term

$$\begin{aligned}
&\int_{x_1}^{x_2} dx \frac{df(y,y',x)}{dy'}\bigg|_{y=\bar{y}} \frac{d}{dx}\big(y(x) - \bar{y}(x)\big) \\
&= \frac{df(y,y',x)}{dy'}\bigg|_{y=\bar{y}} \big(y(x) - \bar{y}(x)\big)\bigg|_{x=x_1}^{x=x_2} - \int_{x_1}^{x_2} dx \frac{d}{dx} \frac{df(y,y',x)}{dy'}\bigg|_{y=\bar{y}} \big(y(x) - \bar{y}(x)\big)
\end{aligned} \tag{3.62}$$

Since by assumption $y(x_1) = y_1$ and $y(x_2) = y_2$ for any $y(x)$ the non-integral term vanishes and we get

$$\begin{aligned}
&\frac{dI(\alpha)}{d\alpha}\bigg|_{\alpha=0} \\
&= \int_{x_1}^{x_2} dx \bigg[\frac{df(y,y',x)}{dy}\bigg|_{y=\bar{y}} - \frac{d}{dx} \frac{df(y,y',x)}{dy'}\bigg|_{y=\bar{y}} \bigg] \big(y(x) - \bar{y}(x)\big)
\end{aligned} \tag{3.63}$$

Now comes the central point: since $y(x)$ is arbitrary (modulo $y(x_1) = y_1$ and $y(x_2) = y_2$ conditions), the integrand in Eq. (3.63) should vanish identically so we get the condition for $\bar{y}(x)$ to be a stationary point of the functional (3.57) in the form

$$\frac{d}{dx} \frac{df(y,y',x)}{dy'}\bigg|_{y=\bar{y}} \;=\; \frac{df(y,y',x)}{dy}\bigg|_{y=\bar{y}} \tag{3.64}$$

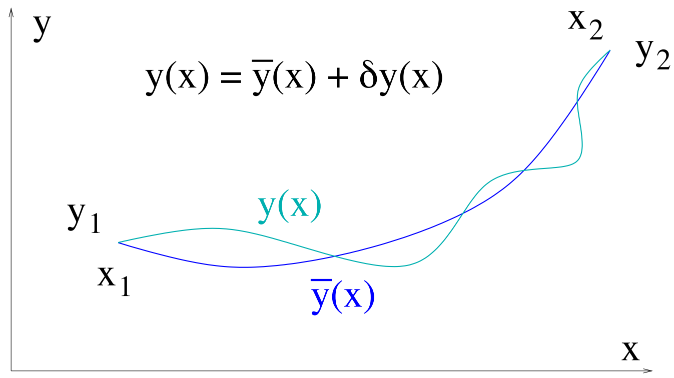

Figure 46. Variation of the path $\bar{y}(x)$

One can represent the derivation of Eq. (3.64) in a more formal way using calculus of variations. We introduce small variations of the path $\bar{y}(x)$

$$\delta y(x) \;=\; y(x) - \bar{y}(x), \qquad\qquad \delta y(x_1) = \delta y(x_2) = 0 \tag{3.65}$$

The variation of the functional (3.57) is then

$$\begin{aligned}\delta I \;&\equiv I(\bar{y}+\delta y, \bar{y}'+\delta y', x) - I(y, y', x) = \int_{x_1}^{x_2} dx \big[f(\bar{y}+\delta y, \bar{y}'+\delta y', x) - f(\bar{y}, \bar{y}', x)\big] \\ &= \int_{x_1}^{x_2} dx \Big\{\frac{\partial f(y,y',x)}{\partial y}\Big|_{y=\bar{y}}\delta y(x) + \frac{\partial f(y,y',x)}{\partial y'}\Big|_{y=\bar{y}}\delta y'(x)\Big\}\end{aligned} \tag{3.66}$$

Integrating by parts the second term we get (cf. Eq. (3.62))

$$\begin{aligned}&\int_{x_1}^{x_2} dx \frac{\partial f(y,y',x)}{\partial y'}\Big|_{y=\bar{y}}\frac{d}{dx}\delta y(x) \\ &= \;-\int_{x_1}^{x_2} dx\; \delta y(x)\frac{d}{dx}\frac{\partial f(y,y',x)}{\partial y'}\Big|_{y=\bar{y}} \;+\; \frac{\partial f(y,y',x)}{\partial y'}\Big|_{y=\bar{y}}\delta y(x)\Big|_{x=x_1}^{x=x_2}\end{aligned} \tag{3.67}$$

By definition (3.65), the variations $\delta y(x)$ vanish at the end points $\delta y(x_1) = \delta y(x_2) = 0$ so the non-integral term in the r.h.s. of Eq. (3.67) vanishes and Eq. (3.66) reduces to

$$\delta I \;=\; \int_{x_1}^{x_2} dx\; \delta y(x)\Big\{\frac{\partial f(y,y',x)}{\partial y}\Big|_{y=\bar{y}} - \frac{d}{dx}\frac{\partial f(y,y',x)}{\partial y'}\Big|_{y=\bar{y}}\Big\} \tag{3.68}$$

Since $\delta y(x)$ is arbitrary, the integrand in the r.h.s. must vanish so we reproduce Eq. (3.64)

3.5.1 Example 1

Q: what function $y(x)$ minimize the distance between points x_1, y_1 and x_2, y_2 on a XY plane?

To answer this question, we need to determine first the functional for this problem. The length of the small segment of the curve Δr is given by (see Fig. 49)

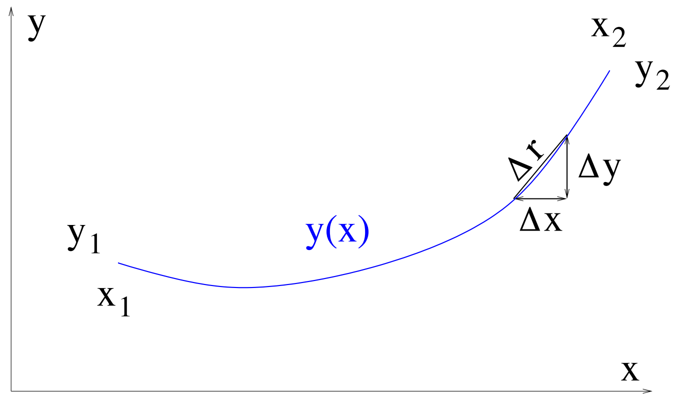

Figure 47. Length of the curve

$$\Delta r \;=\; \sqrt{(\Delta x)^2 + (\Delta y)^2} \;=\; \Delta x\sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}} \tag{3.69}$$

For infinitesimal displacements we get

$$dr \;=\; dx\sqrt{1 + y'^2} \tag{3.70}$$

so the total length between x_1, y_1 and x_2, y_2 is

$$I \;=\; \lim_{(\Delta r)_i \to 0} \sum (\Delta r)_i \;=\; \int_{x_1}^{x_2} dx\, \sqrt{1 + y'^2} \tag{3.71}$$

The Euler-Lagrange equation reads

$$\frac{\partial f}{\partial y} \;=\; 0 \;\Rightarrow\; \frac{d}{dx}\frac{\partial f}{\partial y'} \;=\; \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} \;=\; \frac{y''}{(1 + y'^2)^{3/2}} \;=\; 0 \tag{3.72}$$

so we have

$$y'' \;=\; 0 \;\Rightarrow\; y' = \text{const} \;=\; A \;\Rightarrow\; y \;=\; Ax + B \tag{3.73}$$

This constants A and B can be found from the conditions $y(x_1) = y_1$ and $y(x_2) = y_2$:

$$A \;=\; \frac{y_2 - y_1}{x_2 - x_1}, \quad B \;=\; \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1} \;\Rightarrow\; y \;=\; y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{3.74}$$

Thus, the shortest path between two points on a plane is a straight line.

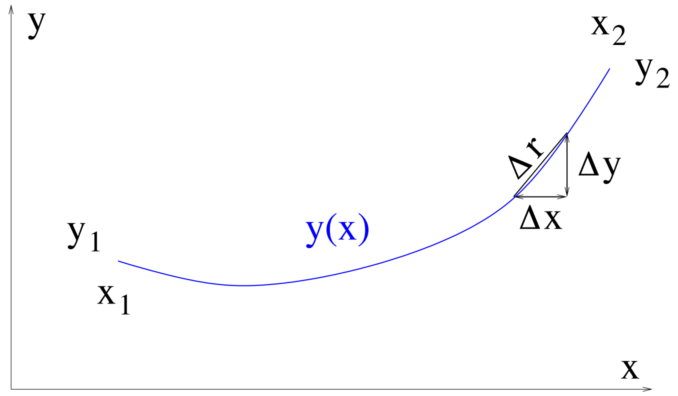

Figure 48. Length of the curve

3.5.2 Example 2

Consider a particle sliding down some slope from point $x = y = 0$ to x_2, y_2 in a uniform gravity field. Find the form of the slope which minimizes the time of the slide. From the previous example: the length of the infinitesimal segment of the path is

$$dr \;=\; dx\,\sqrt{1+y'^2} \tag{3.75}$$

Now, due to conservation of energy the velocity at the point x, y is

$$\frac{mv^2}{2} - gy \;=\; 0 \qquad \Rightarrow \quad v \;=\; \sqrt{\frac{2gy}{m}} \;=\; \frac{dr}{dt} \tag{3.76}$$

and we get

$$dt \;=\; \frac{dr}{v} \;=\; dx\sqrt{\frac{m}{2g}}\sqrt{\frac{y}{1+y'^2}} \qquad \Rightarrow \quad T \;=\; \sqrt{\frac{m}{2g}}\int_0^{x_2} dx\,\sqrt{\frac{1+y'^2}{y}} \tag{3.77}$$

Euler-Lagrange equation

$$y'' \;=\; -\,\frac{1+y'^2}{2y} \tag{3.78}$$

The solution is

$$x \;=\; a(\theta - \sin\theta), \qquad y \;=\; a(1-\cos\theta) \tag{3.79}$$

which is a cycloid $(x - a\theta)^2 + (y - a)^2 \;=\; a^2$.

3.5.3 Variational principle for a functional of many variables

These ideas can be generalized to a functional $I(f)$ of the form

$$I(f) \;=\; \int_{x_1}^{x_2} f\big(y_1(x), y_2(x), ...y_N(x); y_1'(x), y_2'(x), ...y_N'(x), x\big) \tag{3.80}$$

Repeating the steps (3.66) -(3.67) we get

$$\begin{aligned}
&\int_{x_1}^{x_2} f\big(y_1(x)+\delta y_1(x), y_2(x), ...y_N(x); y_1'(x)+\delta y_1'(x), y_2'(x), ...y_N'(x), x\big) \\
&-\int_{x_1}^{x_2} f\big(y_1(x), y_2(x), ...y_N(x); y_1'(x), y_2'(x), ...y_N'(x), x\big) \\
&= \text{ same tricks } = \int_{x_1}^{x_2} dx\ \delta y_1(x)\Big(\frac{\partial f}{\partial y_n} - \frac{d}{dx}\frac{\partial f}{\partial y_n'}\Big)
\end{aligned} \tag{3.81}$$

$\Rightarrow$ Lagrange equations:

$$\frac{\partial f}{\partial y_n} = \frac{d}{dx}\frac{\partial f}{\partial y_n'}, \qquad n = 1, 2, ...N \tag{3.82}$$

Part IX

3.6 Hamilton's principle

Suppose a particle moves along the trajectory $q_i = \bar{q}_i(t)$ between points $q_i(t_1) = q_i^{(1)}$ and $q_i(t_2) = q_i^{(2)}$ ($i = 1, ...N$ - generalized coordinates). Consider any "virtual path" $q_i(t)$ with the same initial and final points and define the "action"

$$S(q(t)) \equiv \int_{t_1}^{t_2} dt\ L(q_i(t), \dot{q}_i(t), t), \qquad L(q_i(t), \dot{q}_i(t), t) = T - V \tag{3.83}$$

Hamilton's principle: from all virtual trajectories with the same initial and final points, the actual path has the least action.

Euler-Lagrange equations for minimum of the action coincide with the Lagrange equations (3.31) which we derived from Newton's laws. Indeed, relabeling $x \to t$ and $y_i(x) \to q_i(t)$ in Eq. (3.80) we get the condition for the extremum of the functional (**??**) in the form

$$\frac{\partial L(q_i(t), \dot{q}_i(t), t)}{\partial q_i}\Big|_{q_i=\bar{q}_i, \dot{q}_i=\dot{\bar{q}}_i} = \frac{d}{dt}\frac{\partial L(q_i(t), \dot{q}_i(t), t)}{\partial \dot{q}_i}\Big|_{q_i=\bar{q}_i, \dot{q}_i=\dot{\bar{q}}_i} \tag{3.84}$$

The Hamilton principle is equivalent to Newton's laws: one could have started classical mechanics course from the statement that for any system there is a function of generalized coordinates $L(q_i(t), \dot{q}_i(t), t)$ such that the system moves along the trajectory with minimal action $S(q(t)) = \int_{t_1}^{t_2} dt\ L(q_i(t), \dot{q}_i(t), t)$.

NB: Note that one can add to the Lagrangian the total derivative of some function (with respect to time) and the Euler-Lagrange equations ($\Rightarrow$ Newton's laws) will not change. Indeed, if $\tilde{L}(q_i(t), \dot{q}_i(t), t) = L(q_i(t), \dot{q}_i(t), t) + \frac{d}{dt}F(q_i(t), t)$ the new action has the form

$$\tilde{S}(q(t)) = \int_{t_1}^{t_2} dt\ \big[L(q_i(t), \dot{q}_i(t), t) + \frac{d}{dt}F(q_i(t), t)\big] = S(q(t)) + F(t_2) - F(t_1) \tag{3.85}$$

Since the classical path corresponds to minimum of $S(q(t))$ at fixed $q(t_1) = q_1$ and $q(t_2) = q_2$ the extra constant $F(q(t_2), t_2) - F(q(t_1), t_1) = F_2 - F_1$ does not affect the variations so Euler-Lagrange equations remain the same.

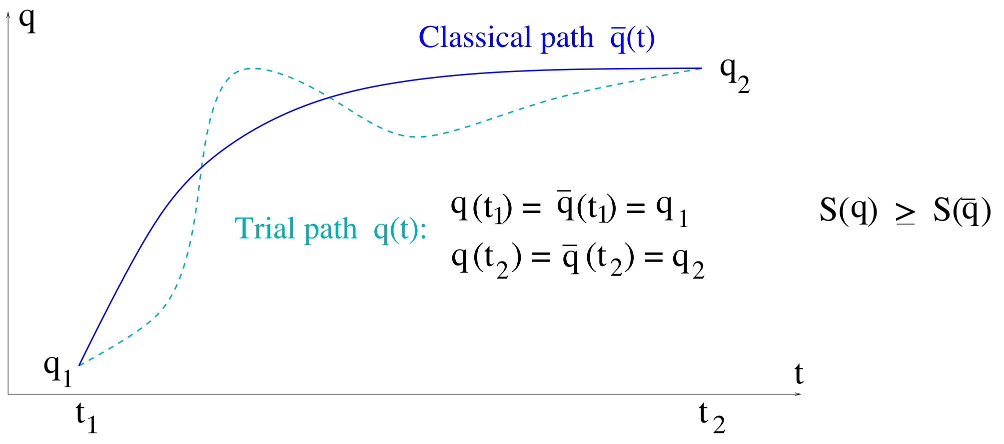

Figure 49. Hamilton principle

it can be also proved directly: the new Euler-Lagrange equations are

$$\begin{aligned}
\frac{\partial}{\partial q_i}L(q_i(t),\dot{q}_i(t),t)+\frac{d}{dt}F(q_i(t),t)\big] \;&=\; \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\big[L(q_i(t),\dot{q}_i(t),t)+\frac{d}{dt}F(q_i(t),t)\big] \Rightarrow \\
\text{l.h.s.} \;&=\; \frac{\partial}{\partial q_i}L(q_i(t),\dot{q}_i(t),t)+\frac{\partial}{\partial q_i}\frac{d}{dt}F(q_i(t),t), \\
\text{r.h.s.} \;&=\; \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}L(q_i(t),\dot{q}_i(t),t)+\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\Big(\dot{q}_k\frac{\partial F(q_i(t),t)}{\partial q_k}+\frac{\partial F(q_i(t),t)}{\partial t}\Big) \\
&=\; \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}L(q_i(t),\dot{q}_i(t),t)+\frac{d}{dt}\frac{\partial}{\partial q_i}F(q_i(t),t)
\end{aligned} \tag{3.86}$$

We need to check that

$$\frac{\partial}{\partial q_i}\frac{d}{dt}F(q_i(t),t) \;=\; \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}F(q_i(t),t) \tag{3.87}$$

Since $\frac{d}{dt}F(q_i(t),t) \;=\; \dot{q}_i(t)\frac{\partial F(q_i(t),t)}{\partial q_i}+\frac{\partial F(q_i(t),t)}{\partial t}$

$$\frac{\partial}{\partial q_i}\frac{d}{dt}F(q_i(t),t)\big] \;=\; \frac{\partial}{\partial q_i}\Big(\dot{q}_k(t)\frac{\partial F(q_i(t),t)}{\partial q_k}+\frac{\partial F(q_i(t),t)}{\partial t}\Big) \;=\; \dot{q}_k(t)\frac{\partial^2 F(q_i(t),t)}{\partial q_i \partial q_k}+\frac{\partial^2 F(q_i(t),t)}{\partial q_i \partial t} \tag{3.88}$$

and

$$\begin{aligned}
\frac{\partial}{\partial q_i}\frac{d}{dt}F(q_i(t),t) \;&=\; \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\Big(\dot{q}_k(t)\frac{\partial F(q_i(t),t)}{\partial q_k}+\frac{\partial F(q_i(t),t)}{\partial t}\Big) \;=\; \frac{d}{dt}\frac{\partial F(q_i(t),t)}{\partial q_i} \\
&=\; \frac{\partial^2 F(q_i(t),t)}{\partial q_i \partial t}+\dot{q}_k\frac{\partial^2 F(q_i(t),t)}{\partial q_k \partial q_i} \;=\; \text{r.h.s. of Eq. (3.88)}
\end{aligned} \tag{3.89}$$

Example: consider $L \;=\; \frac{1}{2}m\dot{\vec{r}}^2$ and $\tilde{L} \;=\; \frac{1}{2}m(\dot{\vec{r}}-\vec{V})^2$ where $\vec{V}$ is some constant vector.

$$\begin{aligned}
\tilde{S} \;&=\; \int_{t_1}^{t_2} dt\frac{1}{2}m(\dot{\vec{r}}-\vec{V})^2 \;=\; \int_{t_1}^{t_2} dt\frac{1}{2}m(\dot{\vec{r}}^2-2\dot{\vec{r}}\cdot\vec{V}+\vec{V}^2) \;=\; \int_{t_1}^{t_2} dt\big[\frac{1}{2}m\dot{\vec{r}}^2+m\frac{d}{dt}(\vec{r}\cdot\vec{V}+\frac{\vec{V}^2}{2}t)\big] \\
&=\; S \;+\; m\big(\vec{r}_2\cdot\vec{V}+\frac{\vec{V}^2}{2}t_2\big)-\big(\vec{r}_1\cdot\vec{V}+\frac{\vec{V}^2}{2}t_1\big)
\end{aligned} \tag{3.90}$$

Euler -Lagrange equation is

$$0 = \frac{\partial \tilde{L}}{\partial r_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{r}_i}\tilde{L} = \frac{d}{dt}(m\dot{\vec{r}} - \vec{V}) = m\ddot{\vec{r}} \tag{3.91}$$

which means that the Lagrangian for a free particle can be written as $\frac{1}{2}mv^2$ in any inertial frame - Newton's law is the same.

3.7 Constants of motion

Let a system have n degrees of freedom, then $q_i \dot{q}_i$ determine uniquely the evolution of the system in time. In general, the positions q_i and velocities $\dot{q}_i$ depend on time. However, there may be certain functions of $f(q_i, \dot{q}_i, t)$ which do not depend on time

$$\frac{d}{dt}f(q_i, \dot{q}_i, t) = 0 \tag{3.92}$$

and then f at any time is determined by initial coordinates and velocities.

For example, if L does not depend on one or more coordinated q_i, the corresponding generalized momentum is conserved

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad \Leftrightarrow \quad p_i = \frac{\partial L}{\partial \dot{q}_i} = \text{const} \tag{3.93}$$

There are three very important conserved quantities related to the property of homogeneity and isotropy of space-time

- i invariance under space translations $\Rightarrow$ conservation of linear momentum,
- ii invariance under rotations $\Rightarrow$ conservation of angular momentum
- iii invariance under time translations $\Rightarrow$ conservation of energy

Note that (i) and (ii) are conservation laws for vector quantities. Depending on the situation, it may be that only one (or two or none) of the components are conserved - if the system is invariant under translations along certain direction or rotations around certain axis.

We will demonstrate now that if the Lagrangian $L(\vec{r}_i, \dot{\vec{r}}_i, t)$ is invariant under these transformations (space translations,rotations, and time translations) the linear momentum, angular momentum, and energy are conserved.

Let us consider a system of N particles and assume that there are no constraints so the Lagrangian can be written as

$$L = \sum_n L(\vec{r}_n, \dot{\vec{r}}_n, t) \tag{3.94}$$

in the Cartesian coordinates.

3.7.1 Space translations

Consider an infinitesimal translation in $\hat{e}_i$ direction: $\vec{r} \to \vec{r} + \vec{\epsilon}$

$$\delta L = \sum_n L(\vec{r}_n + \vec{\epsilon}, \dot{\vec{r}}_n, t) - \sum_n L(\vec{r}_n, \dot{\vec{r}}_n, t) = \sum_n \vec{\epsilon} \cdot \vec{\nabla}^{(n)} L = 0 \quad \Rightarrow \tag{3.95}$$

$$\sum_{n=1}^{N} \vec{\nabla}^{(n)} L = 0 \tag{3.96}$$

where $\vec{\nabla}^{(n)} L = \hat{e}_i \frac{\partial L}{\partial (r_n)_i}$. Next, we use Euler-Lagrange equations and get

$$\frac{\partial L}{\partial (r_n)_i} = \frac{d}{dt} \frac{\partial L}{\partial (\dot{r}_n)_i} \quad \Rightarrow \quad \sum_n \frac{\partial L}{\partial (\dot{r}_n)_i} = \text{const} \tag{3.97}$$

Since $L = \sum_n L(\vec{r}_n, \dot{\vec{r}}_n, t) = \sum \frac{m_n \dot{\vec{r}}_n^2}{2} - V(\vec{r}_n, t)$ we obtain

$$\sum_{n=1}^{N} m_n (\dot{r}_n)_i = \text{const} \quad \Leftrightarrow \quad \sum_{n=1}^{N} m_n \dot{\vec{r}}_n = \sum_{n=1}^{N} \vec{p}_n = \vec{P} = \text{const} \tag{3.98}$$

where $\vec{p} \equiv \sum_{n=1}^{N} \vec{p}_n$ is the total momentum of the set of particles.

3.7.2 Invariance under rotations

Consider a rotation on infinitesimal angle ϵ around axis specified by unit vector $\hat{n}$. From Eq. (2.13) we know that the rotation around the axis defined by $d\vec{\Omega}$ on the angle $|d\vec{\Omega}|$ is represented by the cross product $d\vec{\Omega} \times \vec{r}$ where

$$\vec{r} \to \vec{r}_n' = \vec{r}_n + d\vec{\Omega} \times \vec{r}_n \tag{3.99}$$

where $d\vec{\Omega} = \hat{n}\epsilon$. Also,

$$\dot{\vec{r}} \to \dot{\vec{r}}_n' = \dot{\vec{r}}_n + d\vec{\Omega} \times \dot{\vec{r}}_n \tag{3.100}$$

If we assume $\delta L = 0$ we get

$$\begin{aligned} 0 &= L(\vec{r}_n + d\vec{\Omega} \times \vec{r}_n, \dot{\vec{r}}_n + d\vec{\Omega} \times \dot{\vec{r}}_n, t) - L(\vec{r}_n, \dot{\vec{r}}_n, t) \\ &= \sum_n (d\vec{\Omega} \times \vec{r}_n)_i \frac{\partial L}{\partial (r_n)_i} + \sum_n (d\vec{\Omega} \times \dot{\vec{r}}_n)_i \frac{\partial L}{\partial (\dot{r}_n)_i} \\ &= \sum_n (d\vec{\Omega} \times \vec{r}_n)_i \frac{d}{dt} \frac{\partial L}{\partial (\dot{r}_n)_i} + \sum_n (d\vec{\Omega} \times \dot{\vec{r}}_n)_i \frac{\partial L}{\partial (\dot{r}_n)_i} \\ &= \frac{d}{dt} \sum_n (d\vec{\Omega} \times \vec{r}_n)_i \frac{\partial L}{\partial (\dot{r}_n)_i} = \epsilon \frac{d}{dt} \sum_n (\hat{n} \times \vec{r}_n)_i \frac{\partial L}{\partial (\dot{r}_n)_i} \quad \Rightarrow \quad \frac{d}{dt} \sum_n (\hat{n} \times \vec{r}_n)_i \frac{\partial L}{\partial (\dot{r}_n)_i} = 0 \end{aligned} \tag{3.101}$$

so

$$\sum_n (\hat{n} \times \vec{r}_n)_i \frac{\partial L}{\partial (\dot{r}_n)_i} = \sum_n (\hat{n} \times \vec{r}_n)_i (p_n)_i = \sum_n (\hat{n} \times \vec{r}_n) \cdot \vec{p}_n = \hat{n} \cdot \sum_n \vec{r}_n \times \vec{p}_n = \text{const} \tag{3.102}$$

which means that the component of total angular momentum $\vec{L} = \sum_n \vec{r}_n \times \vec{p}_n$ along $\hat{n}$ direction is conserved.

3.7.3 Invariance under time translations

Consider the time derivative of L (here the constraints may be present)

$$\begin{aligned}\frac{d}{dt}L(q_i,\dot{q}_i,t) &= \dot{q}_i\frac{\partial L}{\partial q_i}+\ddot{q}_i\frac{\partial L}{\partial \dot{q}_i}+\frac{\partial}{\partial t}L \\ &= \dot{q}_i\frac{\partial L}{\partial q_i}+\frac{d}{dt}(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i})-\dot{q}_i\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}+\frac{\partial}{\partial t}L \\ &= \dot{q}_i\frac{\partial L}{\partial q_i}+\frac{d}{dt}(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i})-\dot{q}_i\frac{\partial L}{\partial q_i}+\frac{\partial}{\partial t}L \;=\; \frac{d}{dt}(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i})+\frac{\partial L}{\partial t}\end{aligned} \tag{3.103}$$

where the summation over all coordinates i of all particles is implied. We get

$$\frac{d}{dt}\Big(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L\Big) \;=\; -\frac{\partial L}{\partial t} \tag{3.104}$$

so, if L does not explicitly depend on time ($\equiv \frac{\partial L}{\partial t}=0$), the Hamiltonian

$$H \;=\; \dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L \tag{3.105}$$

is conserved.

If there are only time-independent potentials and time-independent constraints, the Hamiltonian (3.105) is not only constant, but also the total energy. Indeed, the kinetic energy $T = \frac{1}{2}\sum_{n=1}^{N} m_n\dot{r}_n^2$ can be expressed in the generalized corresponds as

$$T \;=\; \sum_{n=1}^{N}\frac{m_n}{2}\sum_{i=1}^{3N-k}\frac{\partial x_n}{\partial q_i}\dot{q}_i\sum_{j=1}^{3N-k}\frac{\partial x_n}{\partial q_j}\dot{q}_j \;=\; \sum_{i=1}^{3N-k}\dot{q}_i\sum_{j=1}^{3N-k}\dot{q}_j\sum_{n=1}^{N}\frac{m_n}{2}\frac{\partial x_n}{\partial q_j}\frac{\partial x_n}{\partial q_i} \tag{3.106}$$

where we used Eq. (3.13) with $\frac{\partial x_i}{\partial t} \;=\; 0$

$$\dot{x}_i \;=\; \sum_{n=1}^{3N-k}\frac{\partial x_i}{\partial q_n}\dot{q}_n+\frac{\partial x_i}{\partial t} \;=\; \sum_{n=1}^{3N-k}\frac{\partial x_i}{\partial q_n}\dot{q}_n \tag{3.107}$$

(here k is a number of constraints). Let us define the symmetric matrix

$$m_{ij} \;\equiv\; \sum_{n=1}^{3N}\frac{m_n}{2}\frac{\partial x_n}{\partial q_i}\frac{\partial x_n}{\partial q_j}, \tag{3.108}$$

then the kinetic energy can be written as

$$T \;=\; \frac{1}{2}\sum_{i,j=1}^{3N-k} m_{ij}\dot{q}_i\dot{q}_j \tag{3.109}$$

Let us compare it to Hamiltonian (3.105). Since the potential energy does not depend on velocities, we get

$$\frac{\partial L}{\partial \dot{q}_i} \;=\; \frac{\partial T}{\partial \dot{q}_i} \;=\; \sum_{j=1}^{3N-k} m_{ij}\dot{q}_j \tag{3.110}$$

and therefore

$$
\begin{aligned}
H &= \sum_{i=1}^{3N-k} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_{i,j=1}^{3N-k} m_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i,j=1}^{3N-k} m_{ij}\dot{q}_i\dot{q}_j + V(q_i) \\
&= \frac{1}{2}\sum_{i,j=1}^{3N-k} m_{ij}\dot{q}_i\dot{q}_j + V(q_i) = T + V(q_i) = E
\end{aligned} \tag{3.111}
$$

When the constraints do depend on time, ther emay be the situation where Hamiltonian (3.105) is conserved but not equal to the total energy.

3.7.4 Number of constants of motion

Consider an isolated system with n degrees of freedom. For isolates system the Lagrangian does not depend on time and so the equations of motion do not involve explicit t-dependence.
Q: How many constants of motion are there?
A: In general, $2n-1$.

Proof: suppose we have solved equations of motion with the initial conditions $q_1(t_0) = c_1, q_2(t_0) = c_2, ...q_n(t_0) = c_n, \dot{q}_1(t_0) = c_{n+1}, ...\dot{q}_n(t_0) = c_{2n}$ and the solution is

$$
\begin{aligned}
q_1 &= q_1(t, c_1, ...c_{2n}),\ q_2 = q_2(t, c_1, ...c_{2n}), ...q_n = q_n(t, c_1, ...c_{2n}), \\
\dot{q}_1 &= \dot{q}_1(t, c_1, ...c_{2n}),\ \dot{q}_2 = \dot{q}_2(t, c_1, ...c_{2n}), ...\dot{q}_n = \dot{q}_n(t, c_1, ...c_{2n})
\end{aligned} \tag{3.112}
$$

Since the system is isolated, it is invariant under time translations so the constants c_i can be rearranged in such a way $C_i = C_i(c_1, ...c_{2n})$ that $C_{2n} = t_0$ so that

$$
\begin{aligned}
q_1 &= q_1(t-t_0, C_1, ...C_{2n-1}),\ q_2 = q_2(t-t_0, C_1, ...C_{2n-1}), , ..., q_n = q_n(t-t_0, C_1, ...C_{2n-1}), , \\
\dot{q}_1 &= \dot{q}_1(t-t_0, C_1, ...C_{2n-1}),\ \dot{q}_2 = \dot{q}_2(t-t_0, C_1, ...C_{2n-1}), ..., \dot{q}_n = \dot{q}_n(t-t_0, C_1, ...C_{2n-1})
\end{aligned} \tag{3.113}
$$

Now we can solve one of these equations, say the last one

$$
t - t_0 = f(\dot{q}_n, C_1, C_2, ...C_{2n-1}) \tag{3.114}
$$

and substitute the obtained $t-t_0$ in the remaining $2n-1$ equations (3.113). We get

$$
\begin{aligned}
q_1 &= q_1(f(\dot{q}_n; \{C_i\}), C_1, ...C_{2n-1}), ...q_{n-1} = q_{n-1}(f(\dot{q}_n; \{C_i\}), C_1, ...C_{2n-1}), q_n = q_n(f(\dot{q}_n; \{C_i\}), C_1, ...C_{2n-1}), \\
\dot{q}_1 &= \dot{q}_1(f(\dot{q}_n; \{C_i\}), C_1, ...C_{2n-1}), ...\dot{q}_{n-1} = (f(\dot{q}_n; \{C_i\}), C_1, ...C_{2n-1})
\end{aligned} \tag{3.115}
$$

At each time t we can solve this system of (2n-1) equations with (2n-1) unknown C_i to get $C_i = F_i\big(q_1(t), ..q_n(t), \dot{q}_1(t), ...\dot{q}_n(t)\big)$. Since C_i are constants, the obtained expressions $F_i\big(q_1(t), ..q_n(t), \dot{q}_1(t), ...\dot{q}_n(t)\big)$ will not depend on time ($\equiv \frac{d}{dt}F_i\big(q_1(t), ..q_n(t), \dot{q}_1(t), ...\dot{q}_n(t)\big) = 0$).

3.7.5 Example: particle in the potential $V(r) = -\frac{\gamma}{r}$.

The Lagrangian is

$$L = \frac{m}{2}\dot{\vec{r}}^2 - \frac{\gamma}{r} \tag{3.116}$$

Since L is time-independent (no constraints are present) the energy is conserved

$$E = H = \frac{m}{2}\dot{\vec{r}}^2 + \frac{\gamma}{r} = \text{const} \tag{3.117}$$

The Lagrangian is invariant under rotations about any axis passing through the center of the force so $\vec{L} = \vec{r}\times\vec{p}$ is conserved. The energy plus 3 components of L give four constants of motion. Moreover, there are three additional constants of motion given by the components of Runge-Lenz vector

$$\vec{A} = \vec{p}\times\vec{L} + m\gamma\hat{r} \tag{3.118}$$

Indeed,

$$\frac{d}{dt}\Big(\vec{p}\times\vec{L} + m\gamma\frac{\vec{r}}{r}\Big) = \dot{\vec{p}}\times\vec{L} + m\gamma\frac{\dot{\vec{r}}}{r} - m\gamma\frac{\vec{r}}{r^2}\dot{r} = \gamma\frac{\vec{r}}{r^3}\times(\vec{r}\times m\dot{\vec{r}}) + m\gamma\Big(\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\dot{r}}{r^2}\Big)$$
$$= \frac{m\gamma}{r^3}[\vec{r}(\vec{r}\cdot\dot{\vec{r}}) - \dot{\vec{r}}r^2] + \frac{m\gamma}{r^3}[\dot{\vec{r}}r^2 - \vec{r}r\dot{r}] = 0 \tag{3.119}$$

where we've used formula

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) \tag{3.120}$$

and the fact that $\vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}\vec{r}^2 = \frac{1}{2}\frac{d}{dt}r^2 = r\dot{r}$.

Thus, it looks like we have $4 = 3 = 7$ constants of motion in contradiction with our theorem which gives $3\times 2 - 1 = 5$ constants. In fact there is no contradiction since not all of our 7 constants are independent; there are two relations among them. The first one is trivial. Since $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$

$$\vec{L}\cdot\vec{A} = \vec{L}\cdot(\vec{p}\times\vec{L}) + m\frac{\gamma}{r}(\vec{L}\cdot\vec{r}) = m\frac{\gamma}{r}\vec{r}\cdot(\vec{p}\times\vec{r}) = 0 \tag{3.121}$$

The second relation is less trivial

$$\vec{A}^2 = (\vec{p}\times\vec{L})^2 + m^2\gamma^2 + 2m\gamma\hat{r}\cdot(\vec{p}\times\vec{L}) \overset{\vec{L}\perp\vec{p}}{=} \vec{p}^2\vec{L}^2 + m^2\gamma^2 + 2m\gamma\hat{r}\cdot(\vec{p}\times\vec{L})$$
$$= \vec{p}^2\vec{L}^2 + m^2\gamma^2 + 2m\gamma(\hat{r}\times\vec{p})\cdot\vec{L} = \big(p^2 + \frac{2m\gamma}{r}\big)L^2 + m^2\gamma^2 = 2mH + m^2\gamma^2 \tag{3.122}$$

Thus, the number of independent constants of motion is $7 - 5 = 2$

Part X

3.8 Forces of constraints

Consider a system described by $3N$ Cartesian coordinates, k holonomic constraints, and hence $3N - k$ degrees of freedom. We can find generalized $3N - k$ coordinates, express the

Lagrangian in terms of these coordinates, and solve the resulting Euler-Lagrange equations. However, this method will tell us nothing about the forces due to the constraints. To find these forces, one needs the method of Lagrange multipliers outlined below.

Suppose we have N particles with coordinates $\vec{r}_1 = (x_1, x_2, x_3)$, $\vec{r}_2 = (x_4, x_5, x_6)$,...$\vec{r}_N = (x_{3N-2}, x_{3N-1}, x_{3N})$ with k constraints

$$\left.\begin{array}{rcl} f_1(x_1, x_2, ...x_{3N}, t) & = & 0 \\ f_2(x_1, x_2, ...x_{3N}, t) & = & 0 \\ & . & \\ & . & \\ & . & \\ f_k(x_1, x_2, ...x_{3N}, t) & = & 0 \end{array}\right\} \quad \text{k constraints, } k \leq 3N \tag{3.123}$$

What we have done before is to solve the equations (3.123) and find the generalized coordinates. However, sometimes it is difficult to solve these equations. Fortunately, there is a trick which enables us to avoid the explicit solution of equations (3.123).

<u>Method of Lagrange multipliers</u>

Consider the system with $3N + k$ generalized coordinates described by the Lagrangian:

$$\begin{aligned} &\tilde{L}(x_1, ..., x_{3N}, \dot{x}_1, ..., \dot{x}_{3N}; \lambda_1, ..., \lambda_k; t) \\ &\quad = \; L(x_1, ..., x_{3N}, \dot{x}_1, ..., \dot{x}_{3N}; t) \; + \; \sum_{j=1}^{k} \lambda_k f_k(x_1, x_2, ...x_{3N}, t) \end{aligned} \tag{3.124}$$

where $x_i, \dot{x}_i$ are our $3N$ original coordinates and $\lambda_1, \lambda_2, ..., \lambda_k$ are k additional coordinates (called Lagrange multipliers).

Hamilton principle for the action

$$\tilde{S} \;=\; \int_{t_1}^{t_2} dt\; \tilde{L}(x_1, ..., x_{3N}, \dot{x}_1, ..., \dot{x}_{3N}; \lambda_1, ..., \lambda_k; t) \tag{3.125}$$

gives us $\delta\tilde{S} = 0$ provided $x_i(t_1)$ and $x_i(t_2)$ are fixed. Considering variations of extremal path $\bar{x}_i(t) \to \bar{x}_i(t) + \delta x_i(t)$ and infinitesimal changes of parameters $\bar{\lambda}_j \to \bar{\lambda}_j + \delta\lambda_j$ we get (cf. Eq. (3.66))

$$\begin{aligned} \delta\tilde{S} \;&=\; \int_{t_1}^{t_2} dt\big[\tilde{L}(\bar{x}_i + \delta x_i, \dot{\bar{x}}_i + \delta\dot{x}_i, \bar{\lambda}_j + \delta\lambda_j) - \tilde{L}(\bar{x}_i, \dot{\bar{x}}_i, \bar{\lambda}_j)\big] \\ &=\; \int_{t_1}^{t_2} dt\Big\{\frac{\partial L}{\partial x_i}\Big|_{x_i=\bar{x}_i}\delta x_i(t) + \frac{\partial\tilde{L}}{\partial\dot{x}_i}\Big|_{x_i=\bar{x}_i}\delta\dot{x}(t) + \sum_{j=1}^{k}\frac{\partial\tilde{L}}{\partial\lambda_j}\delta\lambda_j\Big\} \end{aligned} \tag{3.126}$$

Integrating by parts the second term and using $\delta x_i(t_1) = \delta x_i(t_2) = 0$ we get

$$\begin{aligned} \delta\tilde{S} \;&\equiv I(\bar{y} + \delta y, \bar{y}' + \delta y', x) - I(y, y', x) = \int_{t_1}^{t_2} dt\big[\tilde{L}(\bar{x}_i + \delta x_i, \dot{\bar{x}}_i + \delta\dot{x}_i, \bar{\lambda}_j + \delta\lambda_j) - \tilde{L}(\bar{x}_i, \dot{\bar{x}}_i, \bar{\lambda}_j)\big] \\ =\; &\int_{t_1}^{t_2} dt\Big\{\delta x_i(t)\Big(\frac{\partial\tilde{L}}{\partial x_i}\Big|_{x_i=\bar{x}_i} - \frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{x}_i}\Big|_{x_i=\bar{x}_i}\Big) + \sum_{j=1}^{k}\frac{\partial\tilde{L}}{\partial\lambda_j}\delta\lambda_j\Big\} \end{aligned} \tag{3.127}$$

Thus, Hamilton principle $\delta\tilde{S} = 0$ gives

$$\begin{aligned} \frac{\partial \tilde{L}}{\partial x_i}\Big|_{x_i=\bar{x}_i} &= \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}_i}\Big|_{x_i=\bar{x}_i} \\ \frac{\partial \tilde{L}}{\partial \lambda_j} &= 0 \end{aligned} \tag{3.128}$$

Since $\frac{\partial \tilde{L}}{\partial x_i}\Big|_{x_i=\bar{x}_i} = \frac{\partial L}{\partial x_i}\Big|_{x_i=\bar{x}_i} + \sum_{j=1}^k \lambda_j \frac{\partial f}{\partial x_i}$, $\frac{\partial \tilde{L}}{\partial \dot{x}_i}\Big|_{x_i=\bar{x}_i} = \frac{\partial L}{\partial \dot{x}_i}\Big|_{x_i=\bar{x}_i}$, and $\frac{\partial \tilde{L}}{\partial \lambda_j} = f_j(x_i, t)$ we get [6]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}\Big|_{x_i=\bar{x}_i} - \frac{\partial L}{\partial x_i}\Big|_{x_i=\bar{x}_i} = \sum_{j=1}^k \lambda_j \frac{\partial f_j}{\partial x_i}\Big|_{x_i=\bar{x}_i} \tag{3.129}$$

$$f_j(\bar{x}_1, ..., \bar{x}_{3N}; t) = 0, \qquad j = 1, 2, ...k \tag{3.130}$$

Since $L = T(x_i, \dot{x}_i) - V(x_i)$ we can rewrite Eq.(3.128) as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i}\Big|_{x_i=\bar{x}_i} - \frac{\partial T}{\partial x_i}\Big|_{x_i=\bar{x}_i} = -\frac{\partial V}{\partial x_i}\Big|_{x_i=\bar{x}_i} + \sum_{j=1}^k \lambda_j \frac{\partial f_j}{\partial x_i}\Big|_{x_i=\bar{x}_i} = F_i(\{\bar{x}(t)\}) + R_i(\{\bar{x}(t)\}, t) \tag{3.131}$$

where F_i are forces due to potential $V(x_1, ..x_{3N})$ and

$$R_j \equiv \sum_{j=1}^k \lambda_j \frac{\partial f_j}{\partial x_i}\Big|_{x_i=\bar{x}_i} \tag{3.132}$$

are the forces exerted by the constraints. For example, if the kinetic term is $T = \sum \frac{m_i}{2}\dot{x}_i^2$, the equation (3.131) reads

$$m\ddot{\bar{x}}_i = -\frac{\partial V}{\partial \bar{x}_i} + R_i \tag{3.133}$$

from which it is clear that R_i are additional forces exerted by constraints. Note that in order to find these forces one must solve the equations (3.129), but it is not necessary to solve constraint equations (3.130).

3.8.1 Example

Consider block on the recline.

Constraint: $z = x\tan\alpha$

The Lagrangian (with multipliers) has the form

$$\tilde{L} = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) - mgz + \lambda(z - x\tan\alpha) \tag{3.134}$$

[6] Note that we need to allow time dependence of Lagrange multipliers $\lambda_j = \lambda_j(t)$, otherwise in Eq. (3.128) we will have $\int_{t_1}^{t_2} dt \sum_{j=1}^k \frac{\partial \tilde{L}}{\partial \lambda_j} d\lambda_j = 0$ with time-independent $d\lambda_i$ and the only constraint that we will be able to provide with consntant λ's is $\sum_{j=1}^k d\lambda_j \int_{t_1}^{t_2} dt \frac{\partial \tilde{L}}{\partial \lambda_j} = 0 \Rightarrow \int_{t_1}^{t_2} dt \frac{\partial \tilde{L}}{\partial \lambda_j} = \int_{t_1}^{t_2} dt f_j(\bar{x}_1, ..., \bar{x}_{3N}; t) = 0$ instead of $f_j(\bar{x}_1, ..., \bar{x}_{3N}; t) = 0$ at any time t.

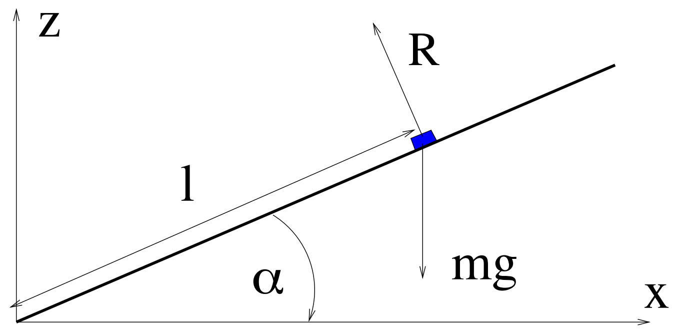

Figure 50. Block on the recline

$$\frac{\partial\tilde{L}}{\partial x} = -\lambda\tan\alpha, \qquad \frac{\partial\tilde{L}}{\partial z} = -mg+\lambda, \qquad \frac{\partial\tilde{L}}{\partial\lambda} = z - x\tan\alpha$$
$$\frac{\partial\tilde{L}}{\partial\dot{x}} = m\dot{x}, \qquad \frac{\partial\tilde{L}}{\partial\dot{z}} = m\dot{z} \tag{3.135}$$

The Euler-Lagrange equations (3.129) take the form

$$\begin{aligned} m\ddot{x} &= -\lambda\tan\alpha \quad\Rightarrow\quad R_x = -\lambda\tan\alpha \\ m\ddot{z} &= -mg+\lambda \quad\Rightarrow\quad R_z = \lambda \\ z &= x\tan\alpha \end{aligned} \tag{3.136}$$

These are 3 equations for 3 unknowns x, z, and λ. Eliminating λ with the help of

$$\lambda = m\ddot{z} + mg = m\ddot{x}\tan\alpha + mg = -\lambda\tan^2\alpha + mg$$

we get

$$\lambda = mg\cos^2\alpha \tag{3.137}$$

Now let us check constraint forces

$$\begin{aligned} R_x &= -mg\sin\alpha\cos\alpha \\ R_z &= mg\cos^2\alpha \end{aligned} \tag{3.138}$$

in accordance with Fig. 50. Now let us find the solutions of equations of motion

$$\begin{aligned} \ddot{x} &= -g\sin\alpha\cos\alpha \quad\Rightarrow\quad x = x_0 - \frac{g}{2}t^2\sin\alpha\cos\alpha \\ \ddot{z} &= -g+g\cos^2\alpha = g\sin^2\alpha \quad\Rightarrow\quad z = z_0 - \frac{g}{2}t^2\sin^2\alpha \end{aligned} \tag{3.139}$$

Note that work done by the reaction force vanishes

$$dW = \vec{R}\cdot d\vec{l} = R_x dx + R_z dz = -mg\sin\alpha\cos\alpha dx + mg\cos^2\alpha dx = 0 \tag{3.140}$$

Alternatively, we could have introduced the generalized coordinate l so the Lagrangian would be

$$L = \frac{m}{2}(\dot{x}^2+\dot{z}^2) - mgz = \frac{m}{2}\dot{l}^2 - mgl\sin\alpha \tag{3.141}$$

The Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{l}} \;=\; \frac{\partial L}{\partial l} \quad \Leftrightarrow \quad m\ddot{l} \;=\; -\,mg\sin\alpha \tag{3.142}$$

and the solution is

$$l \;=\; l_0 - \frac{1}{2}gt^2\sin\alpha \tag{3.143}$$

which is the same as Eq. (3.139) since

$$\sqrt{(x-x_0)^2+(z-z_0)^2} \;=\; \frac{1}{2}gt^2\sin\alpha \;=\; |l-l_0| \tag{3.144}$$

Part XI

4 Small oscillations

4.0.2 Lagrangian for small oscillations: a set of coupled oscillators

In this section we consider the motion of the system undergoing small displacements from a stable equilibrium position. Consider a system of n degrees of freedom described by the generalized coordinates $q_1, q_2, ...q_n$:

$$\left\{\begin{array}{lcl} x_1 & = & x_1(q_1, q_2, ...q_n) \\ x_2 & = & x_2(q_1, q_2, ...q_n) \\ & . & \\ & . & \\ & . & \\ x_{3N} & = & x_{3N}(q_1, q_2, ...q_n) \end{array}\right. \qquad n \leq 3N, \qquad \text{no time dependence} \tag{4.1}$$

where x_i are Cartesian coordinates.The Lagrangian in Cartesian coordinates is

$$L = \sum_{i=1}^{3N} \frac{m_i}{2} \dot{x}_i^2 - V(x_1, ...x_{3N}) \tag{4.2}$$

In terms of generalized coordinates $\dot{x}_i = \sum_{\lambda=1}^{n} \frac{\partial x_i}{\partial q_\lambda} \dot{q}_\lambda$ so

$$\begin{aligned} L &= \sum_{i=1}^{3N} \frac{m_i}{2} \sum_{\lambda=1}^{n} \frac{\partial x_i}{\partial q_\lambda} \dot{q}_\lambda \sum_{\sigma=1}^{n} \frac{\partial x_i}{\partial q_\sigma} \dot{q}_\sigma - V(q_1, ...q_n) \\ &= \sum_{\lambda,\sigma=1}^{n} \frac{m_{\lambda\sigma}}{2} \dot{q}_\lambda \dot{q}_\sigma - V(q_1, ...q_n) \end{aligned} \tag{4.3}$$

where

$$m_{\lambda\sigma} = m_{\lambda\sigma}(q_1, ...q_n) = \sum_{i=1}^{3N} m_i \frac{\partial x_i}{\partial q_\lambda} \frac{\partial x_i}{\partial q_\sigma} \tag{4.4}$$

At any equilibrium $\{q_\sigma^0\}$ (stable or unstable) the generalized force vanishes

$$Q_\sigma = - \left. \frac{\partial V}{\partial q_\sigma} \right|_{q_\sigma = q_\sigma^0} = 0 \tag{4.5}$$

For a system with just one degree of freedom, the condition of a stable equilibrium requires $\left.\frac{\partial^2 V(q)}{\partial q^2}\right|_{q=q^0} > 0$, see Fig. 51

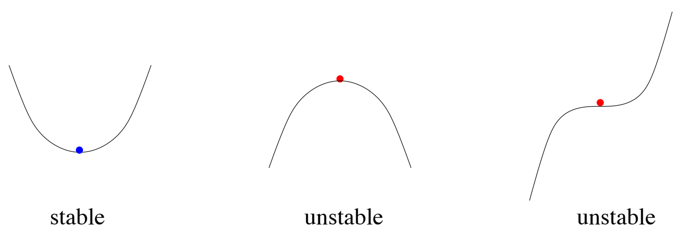

Figure 51. Stable *vs* unstable equilibria

Suppose $q^0_\sigma,\ \sigma = 1, 2, ..n$ is a stable equilibrium. We would like to study small displacements around $\{q^0_\sigma\}$. To this end, we introduce new generalized coordinates η_σ

$$q_\sigma = q^0_\sigma + \eta_\sigma \tag{4.6}$$

and assume that $|\eta_\sigma|$ are small. If $|\eta_\sigma|$ are small, we can expand the potential in Taylor series

$$V(q_1, ...q_n) = V(q^0_1, ...q^0_n) + \frac{1}{2}\sum_{\lambda,\sigma} v_{\lambda,\sigma}\eta_\lambda\eta_\sigma + O(\eta^3), \qquad v_{\lambda\sigma} \equiv \Big(\frac{\partial^2 V}{\partial q_\lambda \partial q_\sigma}\Big)\Big|_{q=q^0} \tag{4.7}$$

Similarly

$$T = \sum_{\lambda,\sigma=1}^{n} \frac{m_{\lambda\sigma}}{2}\dot q_\lambda \dot q_\sigma = \sum_{\lambda,\sigma=1}^{n} \frac{m_{\lambda\sigma}}{2}\dot\eta_\lambda\dot\eta_\sigma + O(\eta^3) \tag{4.8}$$

and the Lagrangian takes the form

$$L = T - V = V(q^0_1, ...q^0_n) + \sum_{\lambda,\sigma=1}^{n} \frac{m_{\lambda\sigma}}{2}\dot\eta_\lambda\dot\eta_\sigma - \frac{1}{2}\sum_{\lambda,\sigma} v_{\lambda\sigma}\eta_\lambda\eta_\sigma + O(\eta^3) \tag{4.9}$$

(the overall additive constant $V(q^0_1, ...q^0_n)$ can be omitted).

Note that the coefficients $v_{\lambda\sigma}$ and $m_{\lambda\sigma}$ are real and symmetric in $\lambda \leftrightarrow \sigma$. We can define real symmetric matrices

$$\mathbf{m} = \begin{pmatrix} m_{11} & m_{12} & ... & m_{1n} \\ m_{21} & m_{22} & ... & m_{2n} \\ . & & & \\ . & & & \\ . & & & \\ m_{n1} & m_{n2} & ... & m_{nn} \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} v_{11} & v_{12} & ... & v_{1n} \\ v_{21} & v_{22} & ... & v_{2n} \\ . & & & \\ . & & & \\ . & & & \\ v_{n1} & v_{n2} & ... & v_{nn} \end{pmatrix} \tag{4.10}$$

In the matrix form the Lagrangian (4.9) reads

$$L = \frac{1}{2}\dot{\boldsymbol\eta}^T \mathbf{m}\dot{\boldsymbol\eta} - \frac{1}{2}\boldsymbol\eta^T\mathbf{v}\boldsymbol\eta \tag{4.11}$$

where $\boldsymbol\eta$

$$\boldsymbol\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ . \\ . \\ . \\ \eta_n \end{pmatrix} \qquad \text{and} \qquad \dot{\boldsymbol\eta} = \begin{pmatrix} \dot\eta_1 \\ \dot\eta_2 \\ . \\ . \\ . \\ \dot\eta_n \end{pmatrix} \tag{4.12}$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\eta_\lambda} = \frac{\partial L}{\partial \eta_\lambda} \qquad \Rightarrow \qquad \sum_{\sigma=1}^{n} m_{\lambda\sigma}\ddot\eta_\sigma = -\sum_{\sigma=1}^{n} v_{\lambda\sigma}\eta_\sigma \tag{4.13}$$

This is a set of coupled second-order differential equations. The solution is specified by initial conditions $\eta_\sigma(t = 0)$ and $\dot\eta_\sigma(t = 0)$, $\sigma = 1, 2, ...n$.

4.0.3 Eigenvalues and eigenvectors

For one degree of freedom the set (4.13) reduces to one harmonic-oscillator equation

$$\ddot{\eta} \;=\; -\,v\eta \tag{4.14}$$

with the solution

$$\eta(t) \;=\; \rho\cos(\omega t + \phi), \qquad \omega = \sqrt{\frac{v}{m}} \tag{4.15}$$

with ρ and ϕ fixed by the initial conditions.

Let us try similar ansatz

$$\eta_\nu(t) \;=\; \rho_\nu \cos(\omega t + \phi), \qquad \nu \;=\; 1, 2, ...n \tag{4.16}$$

for n degrees of freedom. Substituting this ansatz into Eq. (4.13) we get

$$\sum_\sigma m_{\lambda\sigma}\omega^2\rho_\sigma \cos(\omega t + \phi) \;=\; \sum_\sigma v_{\lambda\sigma}\rho_\sigma \cos(\omega t + \phi) \tag{4.17}$$

$\Rightarrow$

$$\sum_\sigma (v_{\lambda\sigma} - m_{\lambda\sigma}\omega^2)\rho_\sigma \;=\; 0, \qquad \lambda \;=\; 1, 2, ...n \tag{4.18}$$

or, in matrix notations,

$$(\mathbf{v} \;-\; \omega^2\mathbf{m})\boldsymbol{\rho} \;=\; 0 \qquad \Leftrightarrow \qquad (\mathbf{m}^{-1}\mathbf{v} - \omega^2)\boldsymbol{\rho} \;=\; 0 \tag{4.19}$$

This is an eigenvalue problem which has solution only if

$$\det|\mathbf{m}^{-1}\mathbf{v} - \omega^2| \;=\; 0 \qquad \Leftrightarrow \qquad \det|\mathbf{v} - \omega^2\mathbf{m}| \;=\; 0 \tag{4.20}$$

The determinant in the r.h.s. of Eq. (4.20) is a polynomial in ω^2 of order n. Any such polynomial has n roots ω_s^2, $\;s \;=\; 1, ..., n$ (some of the roots may coincide). Let us prove that since matrices $\mathbf{m}$ and $\mathbf{v}$ are symmetric and real, all roots ω_s^2 are real.

Proof: take

$$\boldsymbol{\eta}^\dagger(\mathbf{v} - \omega^2\mathbf{m})\boldsymbol{\eta} \;=\; 0 \qquad \Leftrightarrow \qquad \omega^2 \;=\; \frac{\boldsymbol{\eta}^\dagger\mathbf{v}\boldsymbol{\eta}}{\boldsymbol{\eta}^\dagger\mathbf{m}\boldsymbol{\eta}} \tag{4.21}$$

where $\boldsymbol{\eta}^\dagger \equiv \boldsymbol{\eta}^{T*}$. The eigenvectors $\boldsymbol{\eta}$ may be imaginary, but $\boldsymbol{\eta}^\dagger\mathbf{v}\boldsymbol{\eta}$ is real:

$$(\boldsymbol{\eta}^\dagger\mathbf{v}\boldsymbol{\eta})^* \;=\; \big(\boldsymbol{\eta}^{T*}\mathbf{v}\boldsymbol{\eta}\big)^* \;=\; \sum_{\lambda,\sigma}(\eta^*_\lambda v_{\lambda\sigma}\eta_\sigma)^* \;=\; \sum_{\lambda,\sigma}\eta_\lambda v_{\lambda\sigma}\eta^*_\sigma \;=\; \sum_{\lambda,\sigma}\eta^*_\sigma v_{\sigma\lambda}\eta_\lambda \;=\; \boldsymbol{\eta}^{T*}\mathbf{v}\boldsymbol{\eta} \;=\; \boldsymbol{\eta}^\dagger\mathbf{v}\boldsymbol{\eta} \tag{4.22}$$

Similarly, $\mathbf{m}$ is symmetric and real, therefore $\boldsymbol{\eta}^{T*}\mathbf{m}\boldsymbol{\eta}$ is also real and so is ω_s^2 given by the ratio (4.21). We will consider the case when all ω_s^2 are positive [7].

For next step we will need a formula

$$(\boldsymbol{\chi}^\dagger\mathbf{A}\boldsymbol{\xi})^* \;=\; \big(\sum_{\lambda,\sigma}\chi^*_\lambda A_{\lambda\sigma}\xi_\sigma\big)^* \;=\; \sum_{\lambda,\sigma}\chi_\lambda A^*_{\lambda\sigma}\xi^*_\sigma \;=\; \sum_{\lambda,\sigma}\xi^*_\sigma A^\dagger_{\sigma\lambda}\chi_\lambda \;=\; \boldsymbol{\xi}^\dagger\mathbf{A}^\dagger\boldsymbol{\chi} \tag{4.23}$$

[7] If some ω_s^2 are positive and some negative, we have an unstable equilibrium of the saddle-point type

where $A^\dagger_{\sigma\lambda} \equiv A^*_{\lambda\sigma}$. Note that for the symmetric real matrix $\mathbf{A}^\dagger = \mathbf{A}$.

Let us prove now that the eigenvectors corresponding to different eigenvalues are orthogonal with weight $\mathbf{m}$. Consider two eigenvectors $\boldsymbol{\eta}^s$ and $\boldsymbol{\eta}^t$ corresponding to two different eigenvalues ω_s^2 and ω_t^2

$$(\mathbf{v} - \omega_s^2 \mathbf{m})\boldsymbol{\eta}^{(s)} \; = \; 0, \qquad (\mathbf{v} - \omega_t^2 \mathbf{m})\boldsymbol{\eta}^{(t)} \; = \; 0, \tag{4.24}$$

Let us multiply the first equation by $\boldsymbol{\eta}^{(t)^\dagger}$ and second equation by $\boldsymbol{\eta}^{(s)^\dagger}$

$$\boldsymbol{\eta}^{(t)\dagger}(\mathbf{v} - \omega_s^2 \mathbf{m})\boldsymbol{\eta}^{(s)} \; = \; 0, \qquad \boldsymbol{\eta}^{(s)\dagger}(\mathbf{v} - \omega_t^2 \mathbf{m})\boldsymbol{\eta}^{(t)} \; = \; 0, \tag{4.25}$$

The complex conjugate of the first equation is

$$\left(\boldsymbol{\eta}^{(t)\dagger}(\mathbf{v} - \omega_s^2 \mathbf{m})\boldsymbol{\eta}^{(s)}\right)^* \; = \; \boldsymbol{\eta}^{(s)\dagger}(\mathbf{v} - \omega_s^2 \mathbf{m})\boldsymbol{\eta}^{(t)}$$

(see Eq. (4.23). Now, subtracting this equation from the second equation in (4.25) we get

$$\boldsymbol{\eta}^{(t)\dagger}(\omega_t^2 - \omega_s^2)\mathbf{m}\boldsymbol{\eta}^{(s)} \; = \; 0 \tag{4.26}$$

and therefore

$$\boldsymbol{\eta}^{(t)\dagger}\mathbf{m}\boldsymbol{\eta}^{(s)} \; = \; 0 \qquad \text{if} \qquad \omega_s^2 \; \neq \; \omega_t^2 \tag{4.27}$$

Suppose now that all the eigenvalues are different [8]. The corresponding eigenvectors are orthogonal. Moreover, they can be normalized by the condition

$$\boldsymbol{\eta}^{(t)\dagger}\mathbf{m}\boldsymbol{\eta}^{(s)} \; = \; \delta_{st} \tag{4.28}$$

Indeed, since the homogeneous linear system (4.18) has zero determinant, only $n-1$ equations are linearly independent. For example, we can choose first (n-1) equations

$$\begin{cases} (v_{11} - \omega_s^2 m_{11})\rho_1^s + \ldots + (v_{1,n-1} - \omega_s^2 m_{1,n-1})\rho_{n-1}^s + (v_{1n} - \omega_s^2 m_{1n})\rho_n^s = 0 \\ (v_{21} - \omega_s^2 m_{21})\rho_1^s + \ldots + (v_{2,n-1} - \omega_s^2 m_{2,n-1})\rho_{n-1}^s + (v_{2n} - \omega_s^2 m_{2n})\rho_n^s = 0 \\ \cdot \\ \cdot \\ \cdot \\ (v_{n-1,1} - \omega_s^2 m_{n-1,1})\rho_1^s + \ldots + (v_{n-1,,n-1} - \omega_s^2 m_{n-1,n-1})\rho_{n-1}^s + (v_{n-1,n} - \omega_s^2 m_{n-1,n})\rho_n^s = 0 \end{cases} \tag{4.29}$$

and rewrite them as

$$\begin{cases} (v_{11} - \omega_s^2 m_{11})\frac{\rho_1^s}{\rho_n^s} + \ldots + (v_{1,n-1} - \omega_s^2 m_{1,n-1})\frac{\rho_{n-1}^s}{\rho_n^s} = \omega_s^2 m_{1n} - v_{1n} \\ (v_{21} - \omega_s^2 m_{21})\frac{\rho_1^s}{\rho_n^s} + \ldots + (v_{2,n-1} - \omega_s^2 m_{2,n-1})\frac{\rho_{n-1}^s}{\rho_n^s} = \omega_s^2 m_{2n} - v_{2n} \\ \cdot \\ \cdot \\ \cdot \\ (v_{n-1,1} - \omega_s^2 m_{n-1,1})\frac{\rho_1^s}{\rho_n^s} + \ldots + (v_{n-1,,n-1} - \omega_s^2 m_{n-1,n-1})\frac{\rho_{n-1}^s}{\rho_n^s} = \omega_s^2 m_{n-1,n} - v_{n-1,n} \end{cases} \tag{4.30}$$

[8] For degenerate eigenvalues one can get a set of orthonormal eigenvectors using Gram-Schmidt procedure: given the set of linearly independent vectors $\mathbf{v}_n$, choose $\hat{\mathbf{v}}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}$, $\bar{\mathbf{v}}_2 = \mathbf{v}_2 - \hat{\mathbf{v}}_1(\hat{\mathbf{v}}_1 \cdot \mathbf{v}_2)$ and $\hat{\mathbf{v}}_2 = \frac{\bar{\mathbf{v}}_2}{|\bar{\mathbf{v}}_2|}$, $\bar{\mathbf{v}}_3 = \mathbf{v}_3 - \hat{\mathbf{v}}_1(\hat{\mathbf{v}}_1 \cdot \mathbf{v}_3) - \hat{\mathbf{v}}_2(\hat{\mathbf{v}}_2 \cdot \mathbf{v}_3)$ and $\hat{\mathbf{v}}_3 = \frac{\bar{\mathbf{v}}_3}{|\bar{\mathbf{v}}_3|}$, etc. In general, $\bar{\mathbf{v}}_n = \mathbf{v}_n - \sum_{k=1}^{n-1} \hat{\mathbf{v}}_k(\hat{\mathbf{v}}_k \cdot \mathbf{v}_n)$ and $\hat{\mathbf{v}}_n = \frac{\bar{\mathbf{v}}_n}{|\bar{\mathbf{v}}_n|}$. It is easy to see now that the set $\hat{\mathbf{v}}_n$ is orthonormal.

This is $n-1$ equations which determine $n-1$ ratios $\frac{\rho^s_k}{\rho^s_n}$ and therefore all ρ^s_k are determined only up to an overall factor. We can choose this factor in such a way that Eq. (4.28) is satisfied. Moreover, all coefficients in the system (4.30) are real which means that the ratios $\frac{\rho^s_k}{\rho^s_n}$ are real, too. The complexity can enter the solutions η^s only as an overall factor. Thus, we can always write down the solution of the equation $(\mathbf{v} \;-\; \omega_s^2\mathbf{m})\boldsymbol{\eta}_s \;=\; 0$ in the form

$$\boldsymbol{\eta}_s \;=\; C_s e^{i\phi_s}\boldsymbol{\rho}_s \tag{4.31}$$

where $\boldsymbol{\rho}_s$ are real orthonormal vectors

$$\boldsymbol{\rho}^{(t)\dagger}\mathbf{m}\boldsymbol{\rho}^{(s)} \;=\; \delta_{st} \tag{4.32}$$

4.0.4 General solution and initial conditions

To summarize, we have found a set of n independent solutions

$$\boldsymbol{\rho}^{(s)}\cos(\omega_s t+\phi_s) \;=\; \begin{pmatrix} \rho_1^{(s)} \\ \rho_2^{(s)} \\ \cdot \\ \cdot \\ \cdot \\ \rho_n^{(s)} \end{pmatrix} cos(\omega_s t+\phi_s), \qquad s \;=\; 1,2,...n \tag{4.33}$$

with $\boldsymbol{\rho}^{(s)}$ normalized according to

$$\boldsymbol{\rho}^{(t)\dagger}\mathbf{m}\boldsymbol{\rho}^{(s)} \;=\; \delta_{st} \tag{4.34}$$

The general solution of the equation $(\mathbf{v} \;-\; \omega_s^2\mathbf{m})\boldsymbol{\eta}_s \;=\; 0$ can be written as

$$\boldsymbol{\eta}(t) \;=\; \sum_{s=1}^n C_s\boldsymbol{\rho}^{(s)}\cos(\omega_s t+\phi_s) \tag{4.35}$$

where real constants C_s are determined by the initial conditions. To find C_s and ϕ_s, consider

$$\begin{aligned} \boldsymbol{\eta}(0) \;&=\; \sum_{s=1}^n C_s\boldsymbol{\rho}^{(s)}\cos\phi_s \qquad \Rightarrow \qquad \boldsymbol{\rho}^{(r)\dagger}\mathbf{m}\boldsymbol{\eta}(0) \;=\; C_r\cos\phi_r \\ \dot{\boldsymbol{\eta}}(0) \;&=\; -\sum_{s=1}^n C_s\omega_s\boldsymbol{\rho}^{(s)}\sin\phi_s \qquad \Rightarrow \qquad \boldsymbol{\rho}^{(r)\dagger}\mathbf{m}\dot{\boldsymbol{\eta}}(0) \;=\; -\,C_r\omega_r\sin\phi_r \end{aligned} \tag{4.36}$$

and therefore

$$\begin{aligned} \tan\phi_r \;&=\; -\,\frac{1}{\omega_r}\frac{\boldsymbol{\rho}^{(r)\dagger}\mathbf{m}\dot{\boldsymbol{\eta}}(0)}{\boldsymbol{\rho}^{(r)\dagger}\mathbf{m}\boldsymbol{\eta}(0)} \\ C_r^2 \;&=\; \left[\boldsymbol{\rho}^{(r)\dagger}\mathbf{m}\boldsymbol{\eta}(0)\right]^2 + \frac{1}{\omega_r^2}\left[\boldsymbol{\rho}^{(r)\dagger}\mathbf{m}\dot{\boldsymbol{\eta}}(0)\right]^2 \end{aligned} \tag{4.37}$$

The formulas (4.35)-(4.37) determine the solution of Euler-Lagrange equation (4.13).

4.1 Normal modes

4.1.1 Modal matrix

It is convenient to define the modal matrix

$$\mathbf{A}_{\mu\nu} \;\equiv\; \rho_\mu{}^{(\nu)} \;=\; \begin{pmatrix} \rho_1^{(1)} & \rho_1^{(2)} & \dots & \rho_1^{(n)} \\ \rho_1^{(2)} & \rho_2^{(2)} & \dots & \rho_2^{(n)} \\ . \\ . \\ . \\ \rho_n^{(1)} & \rho_n^{(2)} & \dots & \rho_n^{(n)} \end{pmatrix} \tag{4.38}$$

The first (row) index denotes components of the vector $\rho^{(i)}$ and the second (column) index labels different eigenvectors.

Property: **A** diagonalizes both **m** and **v**.

Proof:

$$\begin{aligned}(\mathbf{A}^T\mathbf{m}\mathbf{A})_{\mu\nu} \;&=\; \sum_{\lambda\rho}(A^T)_{\mu\lambda}m_{\lambda\sigma}A_{\rho\nu} \;=\; \sum_{\lambda\sigma}A_{\lambda\mu}m_{\lambda\sigma}A_{\sigma\nu} \;=\; \sum_{\lambda\rho}\rho_\lambda^{(\mu)}m_{\lambda\sigma}\rho_\sigma^{(\nu)} \;=\; \boldsymbol{\rho}^{(\mu)T}\mathbf{m}\boldsymbol{\rho}^{(\nu)} \;=\; \delta_{\mu\nu} \\ (\mathbf{A}^T\mathbf{v}\mathbf{A})_{\mu\nu} \;&=\; \sum_{\lambda\rho}(A^T)_{\mu\lambda}v_{\lambda\sigma}A_{\rho\nu} \;=\; \sum_{\lambda\sigma}A_{\lambda\mu}v_{\lambda\sigma}A_{\sigma\nu} \;=\; \sum_{\lambda\rho}\rho_\lambda^{(\mu)}v_{\lambda\sigma}\rho_\sigma^{(\nu)} \;=\; \boldsymbol{\rho}^{(\mu)T}\mathbf{v}\boldsymbol{\rho}^{(\nu)} \;=\; \omega_\mu^2\delta_{\mu\nu}\end{aligned} \tag{4.39}$$

where we used Eq. (4.34) and Eq. (4.19) so that $(\mathbf{v}-\omega_s^2\mathbf{m})\boldsymbol{\rho}^s = 0 \quad\Rightarrow\quad \boldsymbol{\rho}^{(s)T}\mathbf{v}\boldsymbol{\rho}^{(s)} = \omega_s^2\boldsymbol{\rho}^{(\mu)T}\mathbf{m}\boldsymbol{\rho}^{(\nu)}$.

In matrix notations Eq. (4.39) reads

$$\mathbf{A}^T\mathbf{m}\mathbf{A} \;=\; 1, \qquad\qquad \mathbf{A}^T\mathbf{v}\mathbf{A} \;=\; \begin{pmatrix} \omega_1^2 & 0 & \dots & 0 \\ 0 & \omega_2^2 & \dots & 0 \\ . \\ . \\ . \\ 0 & 0 & \dots & \omega_n^2 \end{pmatrix} \;\equiv\; \boldsymbol{\omega}^2 \tag{4.40}$$

4.1.2 Normal coordinates

Let us introduce a new set of generalized coordinates $\boldsymbol{\xi}(t)$ defined by

$$\boldsymbol{\xi}(t) \;\equiv\; \mathbf{m}\mathbf{A}^T\boldsymbol{\eta}(t) \quad\Leftrightarrow\quad \boldsymbol{\eta}(t) \;=\; \mathbf{A}\boldsymbol{\xi}(t) \tag{4.41}$$

In terms of these new coordinates the Lagrangian (4.11) reduces to the sum of uncoupled oscillators. Indeed,

$$\begin{aligned} L \;&=\; \frac{1}{2}\dot{\boldsymbol{\eta}}^T m\dot{\boldsymbol{\eta}} - \frac{1}{2}\boldsymbol{\eta}^T\mathbf{v}\boldsymbol{\eta} \;=\; \frac{1}{2}\dot{\boldsymbol{\xi}}^T\mathbf{A}^T\mathbf{m}\mathbf{A}\dot{\boldsymbol{\xi}} - \frac{1}{2}\boldsymbol{\xi}^T\mathbf{A}^T\mathbf{v}\mathbf{A}\boldsymbol{\xi} \;=\; \frac{1}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}} - \boldsymbol{\xi}^T\frac{1}{2}\boldsymbol{\omega}^2\boldsymbol{\xi} \;=\; \sum_{\lambda=1}^{n}(\dot{\xi}_\lambda^2 - \omega_\lambda^2\xi_\lambda^2) \\ &=\; \sum_\lambda L_\lambda, \qquad L_\lambda \;=\; \frac{1}{2}\dot{\xi}_\lambda^2 - \frac{1}{2}\omega_\lambda^2\xi_\lambda^2 \end{aligned} \tag{4.42}$$

The normal modes of Lagrangians L_σ are

$$\xi_\lambda = C_\lambda \cos(\omega_\lambda t + \phi_\lambda) \tag{4.43}$$

and the solutions of the original Euler-Lagrange equations (4.13) are

$$\eta_\mu(t) = \sum_{\nu=1}^{n} A_{\mu\nu}\xi_\nu(t) = \sum_{\nu=1}^{n} \rho_\mu^{(\nu)} C_\nu \cos(\omega_\nu t + \phi_\nu) \tag{4.44}$$

where the constants C_ν and ϕ_ν are determined by initial conditions, see Eq. (4.37). Note that the normal coordinates (4.42) are the coefficients of expansion of solution $\boldsymbol{\eta}(t)$ in eigenvectors $\boldsymbol{\rho}^{(\nu)}(t)$ in Eq. (4.44).

Part XII

4.2 Example 1: coupled pendulums

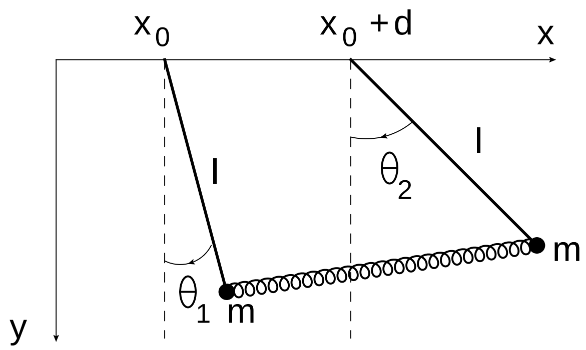

Figure 52. Coupled pendulums

$$\begin{aligned} T &= \frac{m}{2}l^2\dot\theta_1^2 + \frac{m}{2}l^2\dot\theta_2^2 \\ V &= -mgy_1 - mgy_2 + \frac{k}{2}[(x_2 - x_1 - d)^2 + (y_2 - y_1)^2] \end{aligned} \tag{4.45}$$

where

$$\begin{aligned} x_1 &= x_0 + l\sin\theta_1, \qquad & y_1 &= l\cos\theta_1 \\ x_2 &= x_0 + d + l\sin\theta_2, \qquad & y_2 &= l\cos\theta_2 \end{aligned} \tag{4.46}$$

For small displacements $\xi_i = \theta_i$ we get

$$\begin{aligned} x_1 &\simeq x_0 + l\theta_1, \qquad & y_1 &\simeq l - \frac{l}{2}\theta_1^2 \\ x_2 &\simeq x_0 + d + l\theta_2, \qquad & y_2 &\simeq l - \frac{l}{2}\theta_2^2 \end{aligned} \tag{4.47}$$

so

$$V \simeq -2mgl - \frac{m}{2}gl(\theta_1^2+\theta_2^2) + \frac{k}{2}l^2(\theta_1-\theta_2)^2 + O(\theta^4) \tag{4.48}$$

The Lagrangian in generalized coordinates $\xi_i = \theta_i$ takes the form

$$L = \frac{m}{2}l^2(\dot\eta_1^2+\dot\eta_2^2) - \frac{m}{2}gl(\eta_1^2+\eta_2^2) - \frac{kl^2}{2}(\eta_1-\eta_2)^2 \tag{4.49}$$

where we have omitted the overall constant $-2mgl$. The matrices $\mathbf{m}$ and $\mathbf{v}$ are 2×2 matrices

$$\mathbf{m} = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} mgl+kl^2 & -kl^2 \\ -kl^2 & mgl+kl^2 \end{pmatrix} \tag{4.50}$$

and the vectors $\boldsymbol{\eta}$ are two-dimensional

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \dot{\boldsymbol{\eta}} = \begin{pmatrix} \dot\eta_1 \\ \dot\eta_2 \end{pmatrix} \tag{4.51}$$

In matrix notations the Lagrangian (4.49) has the Eq. (4.11) form:

$$L = \frac{1}{2}\dot{\boldsymbol{\eta}}^T\mathbf{m}\dot{\boldsymbol{\eta}} - \frac{1}{2}\boldsymbol{\eta}^T\mathbf{v}\boldsymbol{\eta} \tag{4.52}$$

To get the eigenvalues we must solve the characteristic equation $\det|\mathbf{v}-\omega^2\mathbf{m}| = 0$

$$\det|\mathbf{v}-\omega^2\mathbf{m}| = 0 \quad \Leftrightarrow \quad \det\begin{vmatrix} mgl+kl^2-\omega^2ml^2 & -kl^2 \\ -kl^2 & mgl+kl^2-\omega^2ml^2 \end{vmatrix} = 0 \tag{4.53}$$

which gives

$$(mgl+kl^2-\omega^2ml^2)^2 - k^2l^4 = 0 \quad \Rightarrow \quad mgl+kl^2-\omega^2ml^2 = \pm kl^2 \tag{4.54}$$

Thus, the two possible eigenfrequencies are

$$\omega_1^2 = \frac{g}{l}, \qquad \omega_2^2 = \frac{g}{l} + \frac{2k}{m} \tag{4.55}$$

let us now determine eigenvector for ω_1. The equation is Eq. (4.57)

$$(\mathbf{v}-\omega_1^2\mathbf{m})\boldsymbol{\rho}^{(1)} = 0 \;\Leftrightarrow\; \begin{pmatrix} mgl+kl^2-\omega^2ml^2 & -kl^2 \\ -kl^2 & mgl+kl^2-\omega^2ml^2 \end{pmatrix}\begin{pmatrix} \rho_1^{(1)} \\ \rho_2^{(1)} \end{pmatrix} = 0 \tag{4.56}$$

which gives

$$\begin{aligned} (mgl+kl^2-\omega^2ml^2)\rho_1^{(1)} - kl^2\rho_2^{(1)} &= 0 \\ -kl^2\rho_1^{(1)} + (mgl+kl^2-\omega^2ml^2)\rho_2^{(1)} &= 0 \end{aligned} \tag{4.57}$$

Both equations are satisfied if $\rho_1^{(1)} = \rho_2^{(1)}$ so

$$\boldsymbol{\rho}^{(1)} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.58}$$

This is the mode when two pendulums are in phase.

Similarly, one obtains

$$\boldsymbol{\rho}^{(2)} \;=\; c_2\begin{pmatrix}1\\-1\end{pmatrix} \tag{4.59}$$

This is the mode when the two pendulums oscillate in opposite phase.

To find c_i we use the equation (4.34) $\boldsymbol{\rho}^{(i)\dagger}\mathbf{m}\boldsymbol{\rho}^{(i)} \;=\; 1$. For $i = 1$ one gets

$$c_1^2(1,1)\begin{pmatrix}ml^2 & 0\\ 0 & mgl^2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \;=\; 2ml^2c_1^2 \;=\; 1 \quad\Rightarrow\quad c_1 \;=\; \frac{1}{l\sqrt{2m}} \tag{4.60}$$

Similarly

$$c_2^2\;(1,-1)\begin{pmatrix}ml^2 & 0\\ 0 & mgl^2\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} \;=\; 2ml^2c_2^2 \;=\; 1 \quad\Rightarrow\quad c_2 \;=\; \frac{1}{l\sqrt{2m}} \;=\; c_1 \tag{4.61}$$

Let us find now modal matrix and normal coordinates

$$\mathbf{A} \;=\; \frac{1}{l\sqrt{2m}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{4.62}$$

The normal modes are

$$\boldsymbol{\xi}(t) \;\equiv\; \mathbf{m}\mathbf{A}^T\boldsymbol{\eta}(t) \;=\; \begin{pmatrix}ml^2 & 0\\ 0 & ml^2\end{pmatrix}\frac{1}{l\sqrt{2m}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \;=\; l\sqrt{m/2}\begin{pmatrix}\eta_1+\eta_2\\ \eta_1-\eta_2\end{pmatrix} \tag{4.63}$$

The Lagrangian (4.49) in terms of normal modes reads (see Eq. (4.42))

$$\frac{1}{2}\dot{\boldsymbol{\xi}}^T\dot{\boldsymbol{\xi}} - \boldsymbol{\xi}^T\frac{1}{2}\boldsymbol{\omega}^2\boldsymbol{\xi} = \;\frac{1}{2}\sum_{\lambda=1,2}(\dot{\xi}_\lambda^2 - \omega_\lambda^2\xi_\lambda^2) \;=\; \frac{\dot{\xi}_1^2}{2} - \frac{\omega_1^2}{2}\xi_1^2 + \frac{\dot{\xi}_2^2}{2} - \frac{\omega_2^2}{2}\xi_2^2 \tag{4.64}$$

Check:

$$\begin{aligned}
&\frac{\dot{\xi}_1^2}{2} - \frac{\omega_1^2}{2}\xi_1^2 + \frac{\dot{\xi}_2^2}{2} - \frac{\omega_2^2}{2}\xi_2^2\\
=\;& \frac{ml^2}{4}(\dot{\eta}_1+\dot{\eta}_2)^2 - \frac{gml}{4}(\eta_1+\eta_2)^2 + \frac{ml^2}{4}(\dot{\eta}_1-\dot{\eta}_2)^2 - (\frac{gml}{4}+\frac{kl^2}{2})(\eta_1-\eta_2)^2\\
=\;& \frac{ml^2}{2}(\dot{\eta}_1^2+\dot{\eta}_2^2) - \frac{gml}{2}(\eta_1^2+\eta_2^2) - \frac{kl^2}{2}(\eta_1-\eta_2)^2 \;=\; \text{Eq. (4.49)}
\end{aligned} \tag{4.65}$$

In terms of normal coordinates, the solutions of two uncoupled equations are

$$\xi_i(t) \;=\; C_i\cos(\omega_i t + \phi_i) \tag{4.66}$$

and therefore

$$\eta_i(t) \;=\; \sum_j A_{ij}\xi_j(t) \;=\; \sum \rho_i^{(j)}C_j\cos(\omega_j t + \phi_j) \tag{4.67}$$

or, in explicit form,

$$\eta_1(t) \;=\; \frac{1}{l\sqrt{2m}}(\xi_1(t)+\xi_2(t)), \qquad \eta_2(t) \;=\; \frac{1}{l\sqrt{2m}}(\xi_1(t)-\xi_2(t)) \tag{4.68}$$

As we discussed above, C_i and ϕ_i are determined by the initial conditions. For example, let us take $\eta_1(0) = \alpha$, $\eta_2(0) = \dot{\eta}_1(0) = \dot{\eta}_2(0) = 0$. Using Eq. (4.64) we get

$$\xi_1(0) \;=\; \xi_2(0) \;=\; l\sqrt{\frac{m}{2}}\alpha, \qquad \dot{\xi}_1(0) \;=\; \dot{\xi}_2(0) \;=\; 0 \tag{4.69}$$

and therefore

$$\xi_1(t) \;=\; l\sqrt{\frac{m}{2}}\alpha\cos\omega_1 t, \qquad \xi_2(t) \;=\; l\sqrt{\frac{m}{2}}\alpha\cos\omega_2 t \tag{4.70}$$

From Eq. (4.68) we get

$$\begin{aligned} \eta_1(t) &= \frac{\alpha}{2}(\cos\omega_1 t + \cos\omega_2 t) \\ \eta_1(t) &= \frac{\alpha}{2}(\cos\omega_1 t - \cos\omega_2 t) \end{aligned} \tag{4.71}$$

Using $\cos a + \cos b \;=\; 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$ we can rewrite this in a different way

$$\begin{aligned} \eta_1(t) &= \alpha\big(\cos\frac{\omega_2-\omega_1}{2}t\big)\big(\cos\frac{\omega_2+\omega_1}{2}t\big) \\ \eta_1(t) &= \alpha\big(\sin\frac{\omega_2-\omega_1}{2}t\big)\big(\sin\frac{\omega_2+\omega_1}{2}t\big) \end{aligned} \tag{4.72}$$

Let us consider the case of weak coupling between the oscillators

$$\omega_2 - \omega_1 \ll \omega_2 + \omega_1 \quad \Leftrightarrow \quad \frac{k}{m} \ll \frac{g}{l} \tag{4.73}$$

The formula (4.72) describes rapid oscillations with frequency $\frac{\omega_2+\omega_1}{2}$ and slowly fluctuating amplitude $\sim \cos\frac{\omega_2-\omega_1}{2}t$.

4.3 Example 2: longitudinal waves in one-dimensional crystal

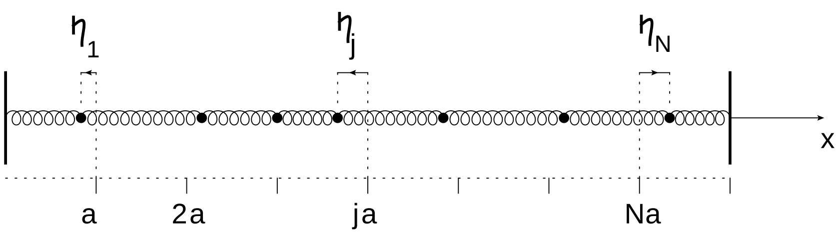

Figure 53. A model for one-dimensional crystal: a set of springs

$a = x^0_{j+1} - x^0_j$ - length of spring when unstretched, $\eta_i \equiv x_j - x^0_j$ - displacement from equilibrium

$$T \;=\; \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + ... + \dot{x}_N^2) \tag{4.74}$$

$$\begin{aligned} V &= \frac{k}{2}(x_1 - a)^2 + \frac{k}{2}(x_2 - x_1 - a)^2 + ... + \frac{k}{2}(x_N - x_{N-1} - a)^2 + \frac{k}{2}(Na + a - x_N)^2 \\ &= \frac{k}{2}\big[\eta_1^2 + \eta_N^2 + \sum_{i=2}^{N}(\eta_i - \eta_{i-1})^2\big] \end{aligned} \tag{4.75}$$

It is convenient to define $\eta_0 \equiv 0 \equiv \eta_{N+1}$, then

$$L = T - V = \frac{m}{2}\sum_{i=1}^{N+1}\dot{\eta}_i^2 - \frac{k}{2}\sum_{i=1}^{N+1}(\eta_i - \eta_{i-1})^2 \tag{4.76}$$

Let us find Euler-Lagrange equations

$$\begin{aligned} \frac{\partial L}{\partial \dot{\eta}_i} &= m\dot{\eta}_i \\ \frac{\partial L}{\partial \eta_i} &= -k(\eta_i - \eta_{i-1}) + k(\eta_{i+1} - \eta_i) = -k(2\eta_i - \eta_{i+1} - \eta_{i-1}) \end{aligned} \tag{4.77}$$

and therefore the equations of motion are

$$m\ddot{\eta}_i = -k(2\eta_i - \eta_{i+1} - \eta_{i-1}) \tag{4.78}$$

which describes "nearest-neighbor" interaction between oscillators.

Define

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ . \\ . \\ . \\ \eta_N \end{pmatrix} \tag{4.79}$$

The matrix $\mathbf{m}$ is trivial

$$\mathbf{m} = m\begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \end{pmatrix} = m\{1\} \tag{4.80}$$

but the matrix $\mathbf{v}$ is not

$$\mathbf{v} = k\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & .. & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & ... & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & ... & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & ... & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & ... & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & ... & -1 & 2 \end{pmatrix} \tag{4.81}$$

The Lagrangian in the matrix form (4.52) is

$$L = \frac{1}{2}\dot{\boldsymbol{\eta}}^T\mathbf{m}\dot{\boldsymbol{\eta}} - \frac{1}{2}\boldsymbol{\eta}^T\mathbf{v}\boldsymbol{\eta} \tag{4.82}$$

Part XIII

4.3.1 Eigenvalues

To find normal coordinates we need to solve the eigenvalue equation $\det|\mathbf{v} - \omega^2\mathbf{m}| \;=\; 0$. It is convenient to define constant $\lambda \equiv \frac{m}{k}\omega^2 - 2$, then

$$\omega^2\mathbf{m} - \mathbf{v} \;=\; k \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & .. & 0 & 0 \\ 1 & \lambda & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & \lambda & 1 & ... & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & \lambda \end{pmatrix} \tag{4.83}$$

To find the determinant of this matrix (up to factor k) we define

$$D^{(N)}(\lambda) \;=\; \det \left.\begin{vmatrix} \lambda & 1 & 0 & 0 & 0 & .. & 0 & 0 \\ 1 & \lambda & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & \lambda & 1 & ... & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & \lambda \end{vmatrix} \qquad \right\} N \tag{4.84}$$

and find the recursion relation between $D^{(N)}(\lambda)$ and $D^{(N-1)}(\lambda)$

$$D^{(N)}(\lambda) \;=\; \lambda \det \underbrace{\begin{vmatrix} \lambda & 1 & 0 & 0 & 0 & .. & 0 & 0 \\ 1 & \lambda & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & \lambda & 1 & ... & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & \lambda \end{vmatrix}}_{N-1} - \det \underbrace{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & .. & 0 & 0 \\ 1 & \lambda & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & \lambda & 1 & ... & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & \lambda \end{vmatrix}}_{N-1}$$

$$= \lambda \det \underbrace{\begin{vmatrix} \lambda & 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 1 & \lambda & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & \lambda & 1 & ... & 0 & 0 \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & \lambda \end{vmatrix}}_{N-1} - \det \underbrace{\begin{vmatrix} \lambda & 1 & 0 & 0 & ... & 0 & 0 \\ 1 & \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & 1 & \lambda & 1 & ... & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & 0 & ... & 1 & \lambda \end{vmatrix}}_{N-2} = \lambda D^{(N-1)}(\lambda) - D^{(N-2)}(\lambda) \tag{4.85}$$

We get the recursion relation

$$D^{(N)}(\lambda) \ = \ \lambda D^{(N-1)}(\lambda) - D^{(N-2)}(\lambda) \tag{4.86}$$

The first two terms are $D^{(1)}(\lambda) \ = \ \lambda$ and

$$D^{(2)}(\lambda) \ = \ \det \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} \ = \ \lambda^2 - 1 \tag{4.87}$$

so

$$\begin{aligned} D^{(3)}(\lambda) \ &= \ \lambda(\lambda^2 - 1) - \lambda \ = \ \lambda^3 - 2\lambda, \\ D^{(4)}(\lambda) \ &= \ \lambda^4 - 3\lambda^2 + 1, \\ D^{(5)}(\lambda) \ &= \ \lambda^5 - 4\lambda^3 + 3\lambda, \\ D^{(6)}(\lambda) \ &= \ ... \end{aligned} \tag{4.88}$$

Ing. guess:

$$D^{(N)}(\lambda) \ = \ A(\lambda) e^{iNB(\lambda)} \tag{4.89}$$

With this ansatz we obtain

$$1 \ = \ \lambda e^{-iB(\lambda)} - e^{-2iB(\lambda)} \quad \Leftrightarrow \quad \cos B(\lambda) \ = \ \frac{\lambda}{2} \tag{4.90}$$

so we must have $|\lambda/2| \leq 1$, otherwise there will be no solution. The equation $\cos B \ = \ \frac{\lambda}{2}$ has two solutions

$$B(\lambda) \ = \ \pm\psi, \quad \psi \ \stackrel{\text{def}}{\equiv} \ \arccos\frac{\lambda}{2} \tag{4.91}$$

The solution for $D^{(N)}(\lambda)$ should be some superposition of two solutions (4.89) with $B(\lambda)$ given by Eq. (4.92):

$$D^{(N)}(\lambda) \ = \ A_+(\lambda) e^{iN\psi} + A_-(\lambda) e^{-iN\psi} \tag{4.92}$$

Since constants $A_\pm(\lambda)$ do not depend on N we can figure them out from the first two determinants

$$\begin{aligned} D^{(1)}(\lambda) \ &= \ \lambda \ = \ 2\cos\psi \ = \ A_+ e^{i\psi} + A_- e^{-i\psi} \\ D^{(2)}(\lambda) \ &= \ \lambda^2 - 1 \ = \ 4\cos^2\psi - 1 \ = \ A_+ e^{2i\psi} + A_- e^{-2i\psi} \end{aligned} \tag{4.93}$$

Solution of the system of two equations with two unknowns

$$
\begin{aligned}
A_+ e^{i\psi} + A_- e^{-i\psi} &= 2\cos\psi \\
A_+ e^{2i\psi} + A_- e^{-2i\psi} &= 4\cos^2\psi - 1
\end{aligned}
\tag{4.94}
$$

gives

$$
A_+ = -\frac{i}{2}\frac{e^{i\psi}}{\sin\psi}, \quad A_- = \frac{i}{2}\frac{e^{-i\psi}}{\sin\psi} = A_+^* \tag{4.95}
$$

and therefore

$$
D^{(N)}(\lambda) = -\frac{i}{2}\frac{e^{i(N+1)\psi}}{\sin\psi} + \frac{i}{2}\frac{e^{-i(N+1)\psi}}{\sin\psi} = \frac{\sin(N+1)\psi}{\sin\psi}, \qquad \psi = \arccos\frac{\lambda}{2} \tag{4.96}
$$

Next, we need to find zeros of our determinant $D^{(N)}(\lambda)$. There are N zeros:

$$
(N+1)\psi_n = \pi m = \pi(N+1-n), \qquad m \text{ (and } n) = 1, 2, ...N \tag{4.97}
$$

so the corresponding eigenfrequencies are

$$
\frac{m}{k}\omega_n^2 - 2 = \lambda_n = 2\cos\psi_n \quad \Rightarrow \quad \omega_n^2 = 4\omega_0^2\cos^2\frac{\psi_n}{2}, \qquad \omega_0^2 = \frac{k}{m} \tag{4.98}
$$

or, in the explicit form

$$
\omega_n = 2\omega_0 \sin\left(\frac{\pi n}{2(N+1)}\right), \qquad \lambda_n = -2\cos\frac{\pi n}{N+1} \tag{4.99}
$$

4.3.2 Eigenvectors

The eigenvectors are the solutions of the equation (4.19)

$$
(\mathbf{v} - \omega^2\mathbf{m})\boldsymbol{\rho} = 0 \quad \Leftrightarrow \quad
\begin{pmatrix}
\lambda_n & 1 & 0 & 0 & 0 & .. & 0 & 0 \\
1 & \lambda_n & 1 & 0 & 0 & ... & 0 & 0 \\
0 & 1 & \lambda_n & 1 & 0 & ... & 0 & 0 \\
0 & 0 & 1 & \lambda_n & 1 & ... & 0 & 0 \\
. & & & & & & & \\
. & & & & & & & \\
. & & & & & & & \\
0 & 0 & 0 & 0 & 0 & ... & \lambda_n & 1 \\
0 & 0 & 0 & 0 & 0 & ... & 1 & \lambda_n
\end{pmatrix}
\begin{pmatrix}
\rho_1^{(n)} \\ \rho_2^{(n)} \\ \rho_3^{(n)} \\ \rho_4^{(n)} \\ . \\ . \\ . \\ \rho_{N-1}^{(n)} \\ \rho_N^{(n)}
\end{pmatrix}
= 0 \tag{4.100}
$$

We get a set of equations

$$
\begin{aligned}
\lambda_n \rho_1^{(n)} + \rho_2^{(n)} &= 0 \\
\rho_1^{(n)} + \lambda_n \rho_2^{(n)} + \rho_3^{(n)} &= 0 \\
\rho_2^{(n)} + \lambda_n \rho_3^{(n)} + \rho_4^{(n)} &= 0 \\
&. \\
&. \\
&. \\
\rho_{N-2}^{(n)} + \lambda_n \rho_{N-1}^{(n)} + \rho_N^{(n)} &= 0 \\
\rho_{N-1}^{(n)} + \lambda_n \rho_N^{(n)} &= 0
\end{aligned}
\tag{4.101}
$$

which is not easy to solve.

A trick: go back to Eq. (4.78) for η's

$$\ddot{\eta}_i \;=\; -\,\omega_0^2(2\eta_i - \eta_{i+1} - \eta_{i-1}), \qquad \omega_0^2 \;=\; \frac{k}{m} \tag{4.102}$$

and try the "traveling wave" ansatz

$$\eta_i \;=\; \Re\big(Ae^{iqx_i^0 - i\omega t}\big) \tag{4.103}$$

where x_i^0 is the equilibrium position of it.h mass Since the equations (4.102) are linear, we can try to find complex solution of the form

$$\eta_i \;=\; Ae^{iqx_i^0 - i\omega t} \tag{4.104}$$

and take the real part in the end of the day.

Substituting the ansatz Eq. (4.104) to the equation of motion (4.102) we obtain

$$\begin{aligned} &-\omega^2 Ae^{iqx_i^0 - i\omega t} + \omega_0^2 e^{-i\omega t}\big[2e^{-iqx_i^0} - e^{-iqx_{i-1}^0} - e^{-iqx_{i+1}^0}\big] \\ &\Leftrightarrow\; \omega^2 \;=\; \omega_0^2\big[2 - e^{qi(x_{i-1}^0 - x_i^0)} - e^{iq(x_{i+1}^0 - x_i^0)}\big] \;=\; 4\omega_0^2 \sin^2\frac{qa}{2} \end{aligned} \tag{4.105}$$

so we get the "dispersion relation"

$$\omega^2 \;=\; 4\omega_0^2 \sin^2\frac{qa}{2} \tag{4.106}$$

Note that the dispersion relation is even in q so the solution of Eq. (4.102) will be a superposition of left-moving and right-moving traveling waves:

$$\eta_i(t) \;=\; A_+ e^{iqx_i^0 - i\omega t} + A_- e^{-iqx_i^0 - i\omega t} \tag{4.107}$$

Next, we need to satisfy "fixed end" boundary conditions

$$\begin{aligned} &\eta_0(t) \;=\; A_+e^{-i\omega t} + A_-e^{-i\omega t} \;=\; 0 \;\;\Rightarrow\;\; A_+ \;=\; -\,A_- \qquad (4.108) \\ &\eta_{N+1}(t) \;=\; A_+e^{iqa(N+1) - i\omega t} + A_-e^{-iqa(N+1) - i\omega t} \;=\; 2iA_+e^{-i\omega t}\sin qa(N+1) \;=\; 0 \end{aligned}$$

which means

$$qa(N+1) \;=\; \pi n \qquad \Leftrightarrow \qquad q_n \;=\; \frac{\pi n}{a(N+1)}, \quad n = 1, 2, ...N \tag{4.109}$$

which is Eq. (4.99). Next, from dispersion relation (4.106) we get

$$\omega_n \;=\; 2\omega_0 \sin\frac{\pi n}{2(N+1)} \tag{4.110}$$

so Eq. (4.107) turns to

$$\eta_j(t) \;=\; 2iA_+e^{-i\omega_n t}\sin q_n x_j^0 \;=\; Be^{-i\omega_n t - i\phi}\sin q_n a j \tag{4.111}$$

where we have redefined $2iA_+ = Be^{-i\phi}$ (with real B and ϕ).

The general solution of Eq. (4.102) is a sum of solutions (4.111) with arbitrary coefficients

$$\eta_j(t) \;=\; \Re\Big\{\sum_{n=1}^{N} B_n e^{-i\omega_n t - i\phi_n}\sin q_n aj\Big\} \;=\; \sum_{n=1}^{N} B_n\cos(\omega_n t+\phi_n)\sin q_n aj \tag{4.112}$$

Now we can return to the system (4.101) and check that

$$\rho_j^{(n)} \;=\; \alpha_n\sin q_n aj \tag{4.113}$$

is a wanted solution corresponding to eigenvalue ω_n. Indeed, for $1<k<N$ we have

$$\begin{aligned}&\rho_{k-1}^{(n)}+\lambda_n\rho_k^{(n)}+\rho_{k+1}^{(n)} \;=\; \alpha_n\big(\sin q_n a(k-1)+\lambda_n\sin q_n ak+\sin q_n a(k+1)\big)\\ &=\; \alpha_n\big(2\sin q_n ak\cos q_n a-2\cos\frac{\pi n}{N+1}\sin q_n ak\big) \;=\; 0\end{aligned} \tag{4.114}$$

(recall that $q_n \;=\; \frac{\pi n}{a(N+1)}$) while for the endpoints

$$\begin{aligned}\lambda_n\rho_1^{(n)}+\rho_2^{(n)} \;&=\; \alpha_n\big(\lambda_n\sin q_n a+\sin 2q_n a\big) \;=\; \alpha_n\big(2\sin q_n a\cos q_n a-2\cos\frac{\pi}{N+1}\sin q_n ak\big) \;=\; 0\\ \rho_{N-1}^{(n)}+\lambda_n\rho_N^{(n)} \;&=\; \alpha_n\big(\sin 2q_n a(N-1)-2\cos\frac{\pi}{N+1}\sin q_n aN\big) \;=\; 0\end{aligned} \tag{4.115}$$

The constants α_n can be found from the normalization condition (4.34)

$$\boldsymbol{\rho}^{(n)\dagger}\mathbf{m}\boldsymbol{\rho}^{(n)} \;=\; 1 \;=\; m\sum_{j=1}^{N}\alpha_n^2\sin^2 q_n aj \;\;\Rightarrow\;\; \alpha_n^2 \;=\; \frac{m^{-1}}{\sum_{j=1}^{N}\sin^2\frac{\pi n}{2(N+1)}j} \tag{4.116}$$

The sum in the denominator can be simplified as:

$$\begin{aligned}&\sum_{j=1}^{N}\sin^2\psi j \;=\; \frac{1}{2}\sum_{j=1}^{N}\big(1-\cos 2\psi j\big) \;=\; \frac{1}{4}\sum_{j=1}^{N}\big(2-e^{2i\psi j}-e^{-2i\psi j}\big)\\ &=\; \frac{N}{2}+\frac{1}{2}-\frac{1}{4}\sum_{j=0}^{N}\big[(e^{2i\psi})^n+(e^{-2i\psi})^n\big] \;=\; \frac{N+1}{2}-\frac{e^{2i\psi(N+1)}-1}{4(e^{2i\psi}-1)}-\frac{e^{-2i\psi(N+1)}-1}{4(e^{-2i\psi}-1)}\end{aligned} \tag{4.117}$$

In our case $\psi \;=\; \frac{\pi n}{2(N+1)}$ so $e^{\pm 2i\psi(N+1)}=1$ and therefore

$$\sum_{j=1}^{N}\sin^2\frac{\pi n}{2(N+1)}j \;=\; \frac{N+1}{2} \;\;\Rightarrow\;\; \alpha_n^2 \;=\; \frac{2}{m(N+1)} \tag{4.118}$$

Thus, the orthonormal set of normal modes is given by Eq. (4.113) with normalization (4.118)

$$\rho_j^{(n)} \;=\; \sqrt{\frac{2}{m(N+1)}}\sin\frac{\pi jn}{(N+1)} \tag{4.119}$$

and the general solution of Eq. (4.113) can be represented as

$$\eta_j(t) \;=\; \sum_{n=1}^{N} C_n\cos(\omega_n t+\phi)\rho_j^{(n)} \tag{4.120}$$

4.4 Example 2a: transverse waves

Consider N identical point masses equally spaced on a stretched massless string with uniform string tension τ.

We will study transverse oscillations in xy plane.

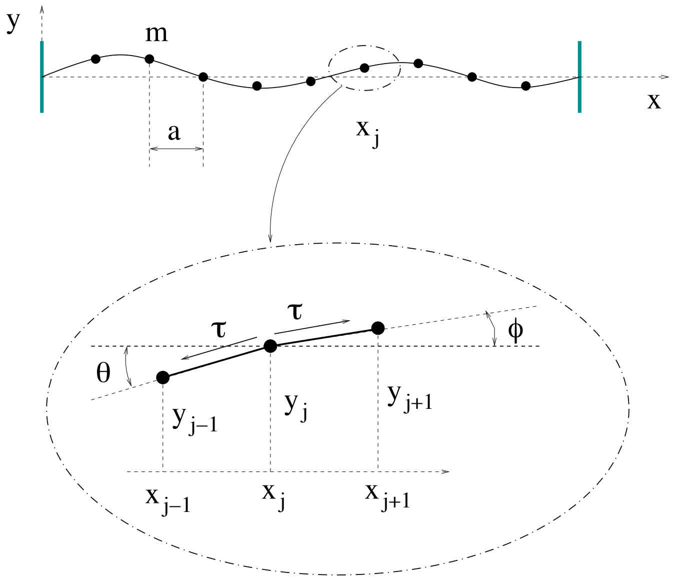

Figure 54. Transverse oscillations

Equation of motion of mass m

$$m\ddot{y} \;=\; \tau\sin\theta - \tau\sin\phi \tag{4.121}$$

We assume that angles are small so $\sin\theta \simeq \tan\theta \simeq \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$ and $\sin\phi \simeq \tan\phi \simeq \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ and the equation of motion (4.120) turns to

$$m\ddot{y}_j \;=\; \frac{\tau}{a}(y_{j+1} - 2y_j + y_{j-1}), \qquad j \;=\; 1, 2, ...N, \qquad y_0 = y_{N+1} = 0 \tag{4.122}$$

This system is governed by the same equation (4.102) as the one-dimensional crystal. The solution is still given by Eq. (4.120)but now it describes *transverse* oscillations:

$$y_j(t) \;=\; \sum_{n=1}^{N} C_n \cos(\omega_n t + \phi)\rho_j^{(n)} \tag{4.123}$$

where eigenvalues ω_n and eigenvectors $\rho_j^{(n)}$ are given by Eqs. (4.110) and (4.119)

$$\omega_n \;=\; 2\sqrt{\frac{\tau}{ma}}\sin\frac{\pi n}{2(N+1)}, \qquad \rho_j^{(n)} \;=\; \sqrt{\frac{2}{m(N+1)}}\sin\frac{\pi j n}{(N+1)} \tag{4.124}$$

Let us introduce

- $L = (N+1)a$ - length of the string
- $c = \sqrt{\frac{\tau a}{m}}$ - characteristic speed of a string

In these terms

$$\omega_n = 2\frac{c}{a}\sin\frac{\pi n a}{2L}, \tag{4.125}$$

The corresponding wavelength is

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2\pi}{\pi n/L} = 2\frac{L}{n} \tag{4.126}$$

Low frequency modes with $n \ll N$ correspond to long wavelengths $\sim L$ and high-frequency modes have wavelength of the order of "lattice spacing" a.

$$\begin{aligned} n \ll N \qquad & \omega_n = \pi n\frac{c}{L} \Rightarrow \omega_{\min} = \frac{\pi c}{L}, \quad \lambda_{\max} = 2L \\ n \to N \qquad & \omega_n \to \omega_{\max} = 2\frac{c}{a}, \qquad \lambda_{\min} = 2a \end{aligned} \tag{4.127}$$

The snapshot (form at a fixed time) of n-th mode of the string

$$y^{(n)}(y,t) = C_n\cos(\omega_n t+\phi)\rho_j^{(n)} \equiv \alpha(t)\rho_j^{(n)} = A(t)\sin\frac{\pi j n a}{L} \tag{4.128}$$

is a sinusoid with $n-1$ knots.

The normal mode amplitudes are propagating wave forms generated by the envelope of the displacements of the particles.

4.5 Continuum limit: non-relativistic string

Continuum limit: $N \to \infty$, $a \to 0$ such that $L = NA =$ fixed. In this limit

$$y_j(t) = y(ja,t) \to y(x,t)$$

where x is the x-coordinate of the string.

The equations of motion in the discrete case are

$$\ddot{y}_j = \frac{\tau}{ma}(y_{j+1}+y_{j-1}-2y_j) \tag{4.129}$$

As $a \to 0$

$$\frac{1}{a}\big[y_{j+1}(t)+y_{j-1}(t)-2y_j(t)\big] = \frac{y_{j+1}(t)-y_j(t)}{a} - \frac{y_j(t)-y_{j-1}(t)}{a} \tag{4.130}$$

$$\begin{aligned} &\frac{1}{a}\big[y_{j+1}(t)+y_{j-1}(t)-2y_j(t)\big] = \frac{y_{j+1}(t)-y_j(t)}{a} - \frac{y_j(t)-y_{j-1}(t)}{a} \\ &= \frac{y(ja+a,t)-y(ja,t)}{a} - \frac{y(ja,t)-y(ja-a,t)}{a} \\ &\stackrel{a\to 0}{\to} \frac{\partial y(x,t)}{\partial x}\Big|_{x=ja+\frac{a}{2}} - \frac{\partial y(x,t)}{\partial x}\Big|_{x=ja-\frac{a}{2}} \simeq a\frac{\partial^2 y(x,t)}{\partial t^2}\Big|_{x=ja} \end{aligned} \tag{4.131}$$

and therefore the equation of motion in the continuum limit is

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\tau}{\sigma}\frac{\partial^2 y(x,t)}{\partial t^2} \tag{4.132}$$

where $\sigma = \frac{m}{a}$ = (constant) mass density. If one rewrites this equation as

$$\frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 y(x,t)}{\partial t^2} \tag{4.133}$$

it becomes the wave equation with sound velocity $c = \sqrt{\frac{\tau a}{m}} = \sqrt{\frac{\tau}{\sigma}}$

4.5.1 Eigenfrequencies and eigenvectors in the continuum limit

Eigenfrequencies in the continuum limit: from Eq. (4.135) we get

$$\omega_n^2 = \frac{2}{a}\sqrt{\frac{\tau}{\sigma}}\sin\frac{\pi n}{2(N+1)} = \frac{2}{a}c\sin\frac{\pi na}{2L} \stackrel{a\to 0}{\to} \frac{\pi nc}{L}, \quad n = 0,1,2,...\infty \tag{4.134}$$

and the eigenvectors are

$$\rho_j^{(n)} = \sqrt{\frac{2}{m(N+1)}}\sin\frac{\pi jn}{(N+1)} \quad \stackrel{a\to 0}{\Rightarrow} \quad \rho^{(n)}(x) = \sqrt{\frac{2}{L\sigma}}\sin\frac{\pi xn}{L} \tag{4.135}$$

Orthogonality of eigenvectors: from Eq. (4.116)

$$\delta_{mn}\sum_{j=1}^{N} m\rho_i^{(n)}\rho_i^{(m)} = \sum_{j=1}^{N} a\frac{m}{a}\rho_i^{(n)}\rho_i^{(m)} \stackrel{a\to 0}{\Rightarrow} \sigma\int_0^L dx\rho^{(m)}(x)\rho^{(n)}(x) \tag{4.136}$$

Next, the general solution of Eq. (4.133) has the form

$$y(x,t) = \sum_{n=1}^{\infty} C_n\cos(\omega_n t+\phi_n)\rho^{(n)}(x) = \sum_{n=1}^{\infty} C_n\cos(\omega_n t+\phi_n)\sqrt{\frac{2}{L\sigma}}\sin\frac{\pi nx}{L} \tag{4.137}$$

with the initial conditions

$$y(x,t=0) = f(x), \qquad \dot{y}(x,t=0) = g(x) \tag{4.138}$$

Expanding in eigenvectors, we get

$$f(x) = \sum C_n\cos\phi_n\rho^{(n)}(x), \qquad g(x) = -\sum C_n\omega_n\cos\phi_n\rho^{(n)}(x) \tag{4.139}$$

Multiplying both sides by $\rho^{(m)}(x)$ and integrating over x we get

$$\sigma\int_0^L dx\ f(x)\rho^{(m)}(x) = C_m\cos\phi_m$$

$$\sigma\int_0^L dx\ g(x)\rho^{(m)}(x) = -C_m\omega_m\cos\phi_m \tag{4.140}$$

From these equations it is easy to determine C_m and ϕ_m.

4.5.2 Lagrangian in the continuum limit

$$L(t) \;=\; \sum_{j=1}^{N}\frac{m_j}{2}\dot{y}_j^2 - \frac{\tau}{2a}\sum_{j=1}^{N}(y_{j+1}-y_j)^2 \;=\; a\sum_{j=1}^{N}\Big[\frac{\sigma}{2}\dot{y}_j^2 - \frac{\tau}{2}\big(\frac{y_{j+1}-y_j}{a}\big)^2\Big] \qquad (4.141)$$

In the continuum limit $\frac{y_{j+1}-y_j}{a} \to y'(x)\Big|_{x=ja}$ and $a\sum_{j=1}^{N} \to \int_0^L dx$ so the Lagrangian (4.141) takes the form

$$L(t) \overset{a\to 0}{=} \int_0^L dx\,\Big[\frac{\sigma}{2}\big(\frac{\partial y}{\partial t}\big)^2 - \frac{\tau}{2}\big(\frac{\partial y}{\partial x}\big)^2\Big] \;=\; \frac{1}{2}\int_0^L dx\,\big[\sigma\dot{y}^2(x,t) - \tau y'^2(x,t)\big] \qquad (4.142)$$

$$\Rightarrow\; L(t) \;=\; \int_0^L dx\,\mathcal{L}(x,t), \qquad \mathcal{L}(x,t) \;=\; \mathcal{L}(y(x,t),\dot{y}(x,t)) \;=\; \frac{1}{2}\big[\sigma\dot{y}^2(x,t) - \tau y'^2(x,t)\big]$$

$\mathcal{L}(x,t)$ is called a Lagrangian density.

The wave equation (4.133) can be obtained from Hamilton's principle: the action with $y(x,t)$ fixed at $t = t_1$ and $t = t_2$ is minimal on classical configuration $\bar{y}(x,t)$. (We assume the boundary condition $y(0,t) \;=\; y(L,t) \;=\; 0$)

The action has the form

$$S \;=\; \int_{t_1}^{t_2} dt\; L(t) \;=\; \int_{t_1}^{t_2} dt \int_0^L dx\;\mathcal{L}(x,t) \qquad (4.143)$$

and the requirement $\delta S = 0$ gives

$$\begin{aligned}
0 \;=\; \delta S \;&=\; \int_{t_1}^{t_2} dt\; L(t) \;=\; \int_{t_1}^{t_2} dt \int_0^L dx\;\delta\mathcal{L}\big(\dot{y}(x,t), y'(x,t)\big)\\
&= \int_{t_1}^{t_2} dt \int_0^L dx\;\Big[\frac{\partial\mathcal{L}(\dot{y},y')}{\partial\dot{y}}\delta\dot{y}(x,t) + \frac{\partial\mathcal{L}(\dot{y},y')}{\partial y'}\delta y'(x,t)\Big]\\
&= \int_{t_1}^{t_2} dt \int_0^L dx\;\Big[\frac{\partial\mathcal{L}(\dot{y},y')}{\partial\dot{y}}\frac{d}{dt}\delta y(x,t) + \frac{\partial\mathcal{L}(\dot{y},y')}{\partial y'}\frac{d}{dx}\delta y(x,t)\Big]\\
&= \int_0^L dx\;\frac{\partial\mathcal{L}(\dot{y},y')}{\partial\dot{y}}\delta y(x,t)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\;\frac{\partial\mathcal{L}(\dot{y},y')}{\partial y}\delta y(x,t)\Big|_0^L\\
&\qquad - \int_{t_1}^{t_2} dt \int_0^L dx\;\delta y(x,t)\Big[\frac{d}{dt}\frac{\partial\mathcal{L}(\dot{y},y')}{\partial\dot{y}} + \frac{d}{dx}\frac{\partial\mathcal{L}(\dot{y},y')}{\partial y'}\Big] \qquad (4.144)
\end{aligned}$$

The requirement of fixed initial and final $y(x,t)$ means $\delta y(x,t_1) \;=\; \delta y(x,t_2) \;=\; 0$ and the boundary requirement $y(0,t) \;=\; y(L,t) \;=\; 0$ gives $\delta y(L,t) \;=\; \delta y(0,t) \;=\; 0$ so we get

$$\begin{aligned}
0 \;&=\; -\int_{t_1}^{t_2} dt \int_0^L dx\;\delta y(x,t)\Big[\frac{d}{dt}\frac{\partial\mathcal{L}(\dot{y},y')}{\partial\dot{y}} + \frac{d}{dx}\frac{\partial\mathcal{L}(\dot{y},y')}{\partial y'}\Big]\\
\Rightarrow\quad & \frac{d}{dt}\frac{\partial\mathcal{L}(\dot{y},y')}{\partial\dot{y}} + \frac{d}{dx}\frac{\partial\mathcal{L}(\dot{y},y')}{\partial y'} \;=\; 0 \qquad \text{Euler}-\text{Lagrange equations} \quad (4.145)
\end{aligned}$$

In our case $\mathcal{L} \;=\; \frac{1}{2}\big[\sigma\dot{y}^2(x,t) - \tau y'^2(x,t)\big]$ so we get wave equation (4.133)

$$\frac{d}{dt}\sigma\dot{y}(x,t) + \frac{d}{dx}(-\tau y'(x,t)) \;=\; 0 \quad\Leftrightarrow\quad \sigma\ddot{y}(x,t) - \tau y''(x,t) \;=\; 0 \qquad (4.146)$$

In general, the Lagrangian density may depend on $\dot{y}$, y', and y

$$S = \int_{t_1}^{t_2} dt\ L(t) = \int_{t_1}^{t_2} dt \int_0^L dx\ \mathcal{L}\big(y(x,t), \dot{y}(x,t), y'(x,t)\big) \tag{4.147}$$

with $y(x,t)$ satisfying some given boundary conditions at end points $x = 0, L$, typically fixed-end conditions $y(0,t) = y(L,t) = 0$ or periodic conditions $y(0,t) = y(L,t),\quad y'(0,t) = y'(L,t)$. In this case we get

$$\begin{aligned}
0 = \delta S = & \int_{t_1}^{t_2} dt\ L(t) = \int_{t_1}^{t_2} dt \int_0^L dx\ \delta\mathcal{L}\big(y(x,t)\dot{y}(x,t), y'(x,t)\big) \qquad (4.148)\\
= & \int_{t_1}^{t_2} dt \int_0^L dx\ \Big[\frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial \dot{y}} \delta\dot{y}(x,t) + \frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial y'} \delta y'(x,t) + \frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial y} \delta y(x,t)\Big]\\
= & \int_{t_1}^{t_2} dt \int_0^L dx\ \Big[\frac{\partial \mathcal{L}(\dot{y},y')}{\partial \dot{y}} \frac{d}{dt}\delta y(x,t) + \frac{\partial \mathcal{L}(\dot{y},y')}{\partial y'} \frac{d}{dx}\delta y(x,t) + \frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial y} \delta y(x,t)\Big]\\
= & \int_0^L dx\ \frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial \dot{y}} \delta y(x,t)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\ \frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial y} \delta y(x,t)\Big|_0^L\\
& + \int_{t_1}^{t_2} dt \int_0^L dx\ \delta y(x,t)\Big[\frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial y} + \frac{d}{dt}\frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial \dot{y}} + \frac{d}{dx}\frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial y'}\Big]
\end{aligned}$$

Again, with fixed-end or periodic boundary conditions and fixed initial and final $y(x,t)$ the non-integral terms in the r.h.s. vanish and we get the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial \dot{y}} + \frac{d}{dx}\frac{\partial \mathcal{L}(y,\dot{y},y')}{\partial y'} \tag{4.149}$$

Part XIV

5 Rigid body dynamics

A rigid body is a special case of a system of particles such that the relative distance between any two particles is fixed. This is clearly an idealization, but a useful one to discuss properties of approximately rigid bodies.

Consider a system of N particles with constraints $r_{ij} = |\vec{r}_{ij}| = c_{ij}$ where c_{ij} are constants. on the first glance, it looks like the system has $\frac{N(N-1)}{2}$ constraints, but not all of them are independent. Let us count the number of degrees of freedom for N particles starting from $N = 3$. For three particles in a general non-collinear positions, we have 3×3=9 coordinates and 3 constraints (r_{12}, r_{23}, r_{13} = fixed). Thus, for 3 particles we have 6 degrees of freedom. Next, for 4 particles we get an extra 3 degrees of freedom for particle #4 but also three new constraints (r_{14}, r_{24}, r_{34} = fixed) so the number of degrees of freedom is still 6. If we add particle #5, we get 3 new coordinates but only 3 new constraints (r_{15}, r_{25}, r_{35} = fixed) since the constraint r_{45} = fixed will be satisfied automatically. Thus, for N=5 we still have 6 degrees of freedom. One can continue adding points and for each new point we have 3 additional coordinates $\vec{r}_n$ and 3 new constraints (r_{1n}, r_{2n}, r_{3n} = fixed). Thus, for arbitrary number of particles with fixed interparticle distances the number of degrees of freedom is 6 which can be identified with the number of translations plus number of rotations of a rigid body.

Three coordinates are needed to specify the origin $\mathcal{O}'$ of the system of coordinates fixed at a point in the rigid body (this point is usually taken in a center of mass), and tree additional coordinates are needed to specify the orientation of the x', y', z' axes fixed in the rigid body relative to a coordinate system with axes parallel to original ones but with the origin at $\mathcal{O}'$.

There are many ways to specify the orientation of x', y', z' axes with respect to original x, y, z axes. For example, one can choose the scalar products of unit vectors specifying the primed and unprimed axes

$$\begin{array}{lllllllll} \hat{e}_x \cdot \hat{e}'_x &=& \cos\theta_{11}, & \hat{e}_x \cdot \hat{e}'_y &=& \cos\theta_{12}, & \hat{e}_x \cdot \hat{e}'_z &=& \cos\theta_{13} \\ \hat{e}_y \cdot \hat{e}'_x &=& \cos\theta_{21}, & \hat{e}_y \cdot \hat{e}'_y &=& \cos\theta_{22}, & \hat{e}_y \cdot \hat{e}'_z &=& \cos\theta_{23} \\ \hat{e}_x \cdot \hat{e}'_x &=& \cos\theta_{31}, & \hat{e}_z \cdot \hat{e}'_y &=& \cos\theta_{32}, & \hat{e}_z \cdot \hat{e}'_z &=& \cos\theta_{33} \end{array} \tag{5.1}$$

Clearly, the 9 parameters $\cos\theta_{mn'}$ are not independent since

$$\begin{array}{lllllll} \hat{e}'_x \cdot \hat{e}'_x &=& \hat{e}'_y \cdot \hat{e}'_y &=& \hat{e}'_z \cdot \hat{e}'_z &=& 1 \\ \hat{e}'_x \cdot \hat{e}'_y &=& \hat{e}'_y \cdot \hat{e}'_z &=& \hat{e}'_z \cdot \hat{e}'_x &=& 0 \end{array} \tag{5.2}$$

so we have 6 constraints

$$\left.\begin{array}{lll} \sum_{n=1}^{3} \cos^2\theta_{mn} &=& 1, \qquad\qquad m = 1,2,3 \\ \sum_{n=1}^{3} \cos\theta_{mn}\cos\theta_{ln} &=& 0, \qquad\qquad m \neq l \end{array}\right\} \quad \Leftrightarrow \quad \sum_{n=1}^{3} \cos\theta_{mn}\cos\theta_{ln} \;=\; \delta_{ml} \tag{5.3}$$

Thus, among 9 different $\cos\theta_{mn}$ we must choose 3 independent parameters which may be functions of θ's. The most common choice is 3 Euler angles introduced below, but first we need to discuss properties of matrix of rotation from x, y, z frame to x', y', z' frame.

Denote $\cos\theta_{mn} \equiv a_{mn}$ and define matrix of rotation

$$\mathbf{A} \;=\; \begin{pmatrix} a_{11}, & a_{12} & a_{13} \\ a_{21}, & a_{22} & a_{23} \\ a_{31}, & a_{32} & a_{33} \end{pmatrix} \tag{5.4}$$

The property (5.3) means that matrix $\mathbf{A}$ is orthogonal

$$\sum_{l=1}^{3} A_{ml}A_{nl} \;=\; \delta_{mn} \quad \Leftrightarrow \quad \mathbf{A}^T\mathbf{A} \;=\; \mathbf{I} \tag{5.5}$$

The matrix $\mathbf{A}$ specifies the rotation from x, y, z frame to x', y', z' frame whereas $\mathbf{A}^T$ describes the rotation back:

$$\begin{pmatrix} \hat{e}'_x \\ \hat{e}'_y \\ \hat{e}'_z \end{pmatrix} \;=\; \mathbf{A}\begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix} \;=\; \mathbf{A}^T\begin{pmatrix} \hat{e}'_x \\ \hat{e}'_y \\ \hat{e}'_z \end{pmatrix} \tag{5.6}$$

One can relate components of any vector $\vec{V}$ in primed and unprimed frames by

$$\begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix} = \mathbf{A} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \qquad \begin{aligned} \vec{V} &= V_1\hat{e}_x + V_2\hat{e}_y + V_3\hat{e}_z \\ &= V_1'\hat{e}_x' + V_2'\hat{e}_y' + V_3'\hat{e}_z' \end{aligned} \tag{5.7}$$

We know that rotations are described by transformations of the Eq. (5.7) type with orthogonal matrices $\mathbf{A}$. For example, (passive) rotation on an angle θ around z axis is given by

$$\left.\begin{aligned} x_1' &= x_1\cos\theta + x_2\sin\theta \\ x_2' &= x_2 - x_1\sin\theta + \cos\theta \\ x_3' &= x_3 \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.8}$$

Summarising, at any time the orientation of a rigid body (x', y', z') relative to external system (x, y, z) is specified by an orthogonal transformation described by an orthogonal matrix $\mathbf{A}(t)$. The 9 elements of this matrix may be expressed in terms of some suitable set of 3 parameters (e.g. Euler angles described below). In general, this orientation changes in time so $\mathbf{A} = \mathbf{A}(t)$.

Let us prove that the determinant $\det|\mathbf{A}(t)| = 1$. Indeed, from $\mathbf{A}(t)^\dagger\mathbf{A}(t) = 1$ we see that $\det|\mathbf{A}(t)| = \pm 1$. If the (x', y', z') frame is chosen to coincide with (x, y, z) frame at $t = 0$ the matrix $\mathbf{A}(0) = \mathbf{I}$ and $\det|\mathbf{A}(0)| = 1$. At a later time, $\mathbf{A}(t)$ might be $\neq \mathbf{I}$, but it must be a *continuous* function of time, which means that $\det|\mathbf{A}(t)$ cannot jump from $+1$ to -1 value.

5.1 Euler angles

We can perform the transformation from $(x, y, z)$to (x', y', z') frame by a sequence of three successive rotations. Each of this rotations is characterized by an angle. It should be noted that the conventions of these three rotations differ in the literature. We will use conventions from our textbook (Fetter & Walecka).

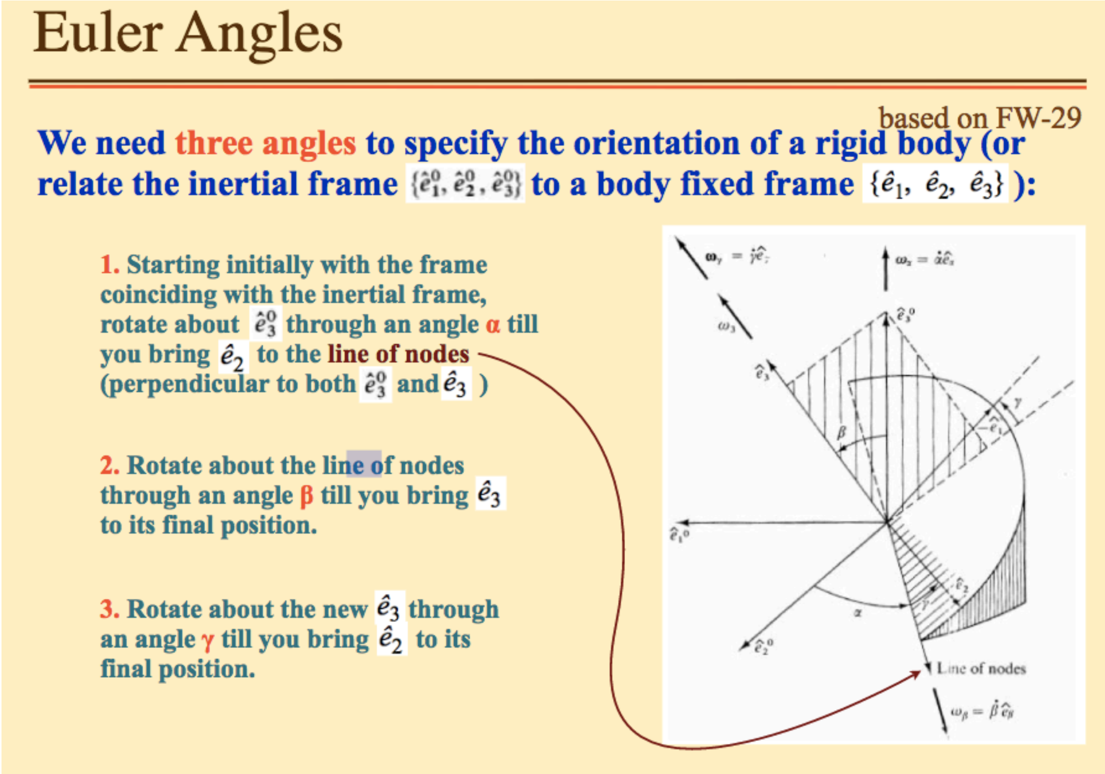

Figure 55. Euler angles

- First rotation $x, y, z \;\rightarrow\; \tilde{x}, \tilde{y}, \tilde{z}$: Rotate by angle α in positive direction (anticlockwise) about z axis bringing $\tilde{y}$ to the orientation denoted as the "line of nodes" (orthogonal to both z and z' axes). The corresponding matrix is

$$\mathbf{D} \;=\; \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \;=\; \mathbf{D} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5.9}$$

- Second rotation $\tilde{x}, \tilde{y}, \tilde{z} \;\rightarrow\; \tilde{x}', \tilde{y}', \tilde{z}'$:
Rotate counterclockwise by angle β about $\tilde{y}$ axis thus bringing $\tilde{z}$ to final orientation $z' = \tilde{z}'$. The corresponding matrix is

$$\mathbf{C} \;=\; \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{z}' \end{pmatrix} \;=\; \mathbf{C} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \tag{5.10}$$

- Third rotation $\tilde{x}', \tilde{y}', \tilde{z}' \;\rightarrow\; x', y', z'$: Rotate counterclockwise by angle γ about $\tilde{z}' = z'$ axis thus bringing $\tilde{y}'$ to final orientation y'. The corresponding matrix is

$$\mathbf{B} \;=\; \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \;=\; \mathbf{B} \begin{pmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{z}' \end{pmatrix} \tag{5.11}$$

After this three rotations

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{BCD} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5.12}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\beta\cos\gamma \\ -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix} \end{aligned} \tag{5.13}$$

Note that

- α and γ varies between 0 and 2π while β varies between 0 and π
- the line of nodes is orthogonal to the plane specified by z and z' axes
- α and β specify the orientation of z' axis relative to (x, y, z) frame.

5.1.1 Angular velocity in terms of Euler angles

The unit vectors in x', y', z' frame are given by the same matrix

$$\begin{pmatrix} \hat{e}'_x \\ \hat{e}'_y \\ \hat{e}'_z \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix} \tag{5.14}$$

(easy to see from $x'\hat{e}'_x + y'\hat{e}'_y + z'\hat{e}'_z = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$) so the vector of angular velocity of the moving frame defined in Eq. (2.17) can be determined from Eq. (2.16) as

$$\begin{aligned} \frac{d\hat{e}'}{dt} &= \vec{\omega} \times \hat{e}' \equiv \frac{d(\hat{e}'_i)_k}{dt} = \epsilon_{klm}\vec{\omega}_l(\hat{e}'_i)_m \Rightarrow \frac{dA_{im}}{dt}(\hat{e}_m)_k = \epsilon_{klm}\vec{\omega}_l(A_{in}\hat{e}_n)_m \\ \Rightarrow \frac{dA_{ik}}{dt} &= \epsilon_{klm}\vec{\omega}_l A_{im} \Rightarrow A^T_{ji}\frac{dA_{ik}}{dt} = \epsilon_{klj}\vec{\omega}_l \Rightarrow \vec{\omega}_l = \frac{1}{2}\epsilon_{lmn}\big(A^T\frac{dA}{dt}\big)_{mn} \end{aligned} \tag{5.15}$$

The matrix $\left(\frac{dA}{dt}A^T\right)$ can be obtained from Eq. (5.13)

$$A^T\frac{dA}{dt} = \mathbf{D}^T\mathbf{C}^T\mathbf{B}^T\dot{\mathbf{B}}\mathbf{C}\mathbf{D} + \mathbf{D}^T\mathbf{C}^T\dot{\mathbf{C}}\mathbf{D} + \mathbf{D}^T\dot{\mathbf{D}} \tag{5.16}$$

Using formula

$$\epsilon_{ijk}M_{im}M_{jn}M_{kl} = \epsilon_{mnl}\det\mathbf{M} \tag{5.17}$$

we see that for any orthogonal matrix $\mathbf{M}$ with $\det\mathbf{M} = 1$

$$\epsilon_{ijk}M_{jm}M_{kn} = M_{il}\epsilon_{mnl} \tag{5.18}$$

and therefore

$$\epsilon_{ijk}\big(\frac{d\mathbf{A}}{dt}\mathbf{A}^T\big)_{jk} \;=\; \epsilon_{imn}(\mathbf{D}^T\dot{\mathbf{D}})_{mn} + \mathbf{D}^T_{il}\epsilon_{lmn}(\mathbf{C}^T\dot{\mathbf{C}})_{mn} + (\mathbf{D}^T\mathbf{C}^T)_{il}\epsilon_{lmn}(\mathbf{B}^T\dot{\mathbf{B}})_{mn} \quad (5.19)$$

Thus, from Eqs. (5.15) and (5.19) we see that $\vec{\omega}$ can be represented as

$$\boldsymbol{\omega} \;=\; \boldsymbol{\omega}^{(\alpha)} + \mathbf{D}^T\boldsymbol{\omega}^{(\beta)} + \mathbf{D}^T\mathbf{C}^T\boldsymbol{\omega}^{(\gamma)} \quad (5.20)$$

where

$$\vec{\omega}_i^{(\alpha)} = \frac{1}{2}\epsilon_{ijk}(\mathbf{D}^T\dot{\mathbf{D}})_{jk} = \begin{pmatrix}0\\0\\\dot{\alpha}\end{pmatrix}, \quad \vec{\omega}_i^{(\beta)} = \frac{1}{2}\epsilon_{ijk}(\mathbf{C}^T\dot{\mathbf{C}})_{jk} = \begin{pmatrix}0\\\dot{\beta}\\0\end{pmatrix}, \quad \vec{\omega}_i^{(\gamma)} = \frac{1}{2}\epsilon_{ijk}(\mathbf{B}^T\dot{\mathbf{B}})_{jk} = \begin{pmatrix}0\\0\\\dot{\gamma}\end{pmatrix} \quad (5.21)$$

Thus, the explicit form of the result (5.20) for vector $\vec{\omega}$ is

$$\vec{\omega} \;=\; \begin{pmatrix}-\dot{\beta}\sin\alpha + \dot{\gamma}\cos\alpha\sin\beta\\ \dot{\beta}\cos\alpha + \dot{\gamma}\sin\alpha\sin\beta\\ \dot{\alpha} + \dot{\gamma}\cos\beta\end{pmatrix} \quad (5.22)$$

5.1.2 Check of $\frac{d\hat{e}'}{dt} \;=\; \vec{\omega}\times\hat{e}'$

$$\begin{aligned}\hat{e}' \;&=\; (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma - \cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma + \cos\alpha\sin\beta)\hat{e}_1\\ &+ (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma - \sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma + \sin\alpha\sin\beta)\hat{e}_2\\ &+ (-\sin\beta\cos\gamma + \sin\beta\sin\gamma + \cos\beta)\hat{e}_3 \quad (5.23)\end{aligned}$$

Explicit check of the 1st component

$$\begin{aligned}(\vec{\omega}\times\hat{e}')_1 \;&=\; [(-\dot{\beta}\sin\alpha + \dot{\gamma}\cos\alpha\sin\beta)\hat{e}_1 + (\dot{\beta}\cos\alpha + \dot{\gamma}\sin\alpha\sin\beta)\hat{e}_2 + (\dot{\alpha} + \dot{\gamma}\cos\beta)\hat{e}_3]\\ &\times [(\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma - \cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma + \cos\alpha\sin\beta)\hat{e}_1\\ &+ (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma - \sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma + \sin\alpha\sin\beta)\hat{e}_2\\ &+ (-\sin\beta\cos\gamma + \sin\beta\sin\gamma + \cos\beta)\hat{e}_3]_1\\ &= (\dot{\beta}\cos\alpha + \dot{\gamma}\sin\alpha\sin\beta)(-\sin\beta\cos\gamma + \sin\beta\sin\gamma + \cos\beta) \quad (5.24)\\ &- (\dot{\alpha} + \dot{\gamma}\cos\beta)(\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma - \sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma + \sin\alpha\sin\beta)\end{aligned}$$

$$\dot{\hat{e}}'_1 \;=\; \frac{d}{dt}(\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma - \cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma + \cos\alpha\sin\beta) \;=\; \text{r.h.s. of Eq. (5.24)}$$

We will need also the components of angular velocity vector $\vec{\omega}$ in the "body" (x', y', z') frame. It has the form

$$\begin{pmatrix}\omega'_1\\\omega'_2\\\omega'_3\end{pmatrix} \;=\; \mathbf{A}\begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix} \;=\; \begin{pmatrix}-\dot{\alpha}\sin\beta\cos\gamma + \dot{\beta}\sin\gamma\\ \dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma\\ \dot{\alpha}\cos\beta + \dot{\gamma}\end{pmatrix} \quad (5.25)$$

Part XV

5.2 Moments of inertia

We have established earlier that the motion of a rigid body with one point fixed is a pure rotation. Denote the inertial frame with the origin somewhere in the body by (x, y, z) and body-fixed frame with the same origin as (x', y', z'). these two frames are connected by the rotation $x'_m = A_{mn}x_n$ with some orthogonal matrix A_{mn} (for example, parametrized by Euler angles as in Eq. (5.13). The kinetic energy for the set of particles is $T = \sum_n \frac{m_n}{2}v_n^2$ (where v's are velocities in the inertial frame). By differentiating $\vec{r} = r_i(t)\hat{e}_i^{(0)} = r'_i(t)\hat{e}'_i(t)$ with respect to time and using Eq. (2.16) we get

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{r}'}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{r} \tag{5.26}$$

(cf. Eq. (2.20)). Now, our rigid body is such system of particles that the all the distances are fixed so $\left(\frac{d\vec{r}'_n}{dt}\right)_{\text{body}} = 0$ and we get

$$T = \sum_n \frac{m_n}{2}\left(\frac{d\vec{r}_n}{dt}\right)^2_{\text{inertial}} = \sum_n \frac{m_n}{2}\left[\vec{\omega}\times\vec{r}_n\right]^2 = \sum_n \frac{m_n}{2}\left[\omega^2 r_n^2 - (\vec{\omega}\cdot\vec{r}_n)^2\right] \tag{5.27}$$

(here we used $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$). It is convenient to rewrite this formula in terms of $(\vec{\omega})_{\text{body}} \equiv \vec{\omega}'$. From Eqs. (5.7) and (5.25) we get

$$\omega^2 = (A^T_{ij}\omega'_j)(A^T_{ik}\omega'_k) = \omega'^2, \qquad \vec{\omega}\cdot\vec{r} = \omega'_j A^T_{jk}A_{ik}r'_k = \vec{\omega}'\cdot\vec{r}' \tag{5.28}$$

and therefore

$$T = \sum_n \frac{m_n}{2}\left[\omega'^2 r'^2_n - (\vec{\omega}'\cdot\vec{r}'_n)^2\right] = \sum_{i,j=1}^{3}\omega'_i\omega'_j\sum_n \frac{m_n}{2}[r'^2_n\delta_{ij} - r'_{ni}r'_{nj}] \tag{5.29}$$

This can be rewritten as $T = \sum_{i,j=1}^3 \omega'_i\omega'_j I_{ij}$ where

$$I'_{ij} = \sum_n \frac{m_n}{2}[r'^2_n\delta_{ij} - r'_{ni}r'_{nj}] \tag{5.30}$$

is called tensor of moments of inertia. Now we see why Eq. (5.29) is more convenient than (5.27) - in the inertial frame I (defined as (5.29) but with r_n's in place of r'_n's) would depend on time!

For the continuous distribution of particles with density $\rho(r')$

$$T = \frac{1}{2}\int d^3r'\rho(r')\left[\omega'^2 r'^2_n - (\vec{\omega}'\cdot\vec{r}'_n)^2\right] = \frac{1}{2}\sum_{i,j}\omega'_i\omega'_j I'_{ij} \tag{5.31}$$

where

$$I' = \int d^3r'\rho(r')[r'^2\delta_{ij} - r'_i r'_j] \tag{5.32}$$

is the tensor of moments of inertia of a rigid body. It depends only on distribution of the mass in the body (and does not depend on time).

5.2.1 Angular momentum of a rigid body

For a set of particles

$$\vec{L} = \sum_n m_n(\vec{r}_n \times \vec{v}_n) = \sum_n m_n(\vec{r}_n \times (\vec{\omega}_n \times \vec{r}_n) = \sum_n m_n[\vec{\omega} r_n^2 - \vec{r}_n(\vec{\omega}\cdot\vec{r}_n)] \tag{5.33}$$

(here we used Eq. (5.25 $\vec{v}_n = \vec{\omega}\times\vec{r}_n$). It is convenient to rewrite this in a fixed-body frame. Since r^2 and $\omega\cdot r$ are scalars we get

$$\vec{L}'_i = \sum_n m_n[\omega'_i {r'_n}^2 - (r'_n)_i(\vec{\omega}'\cdot\vec{r}'_n)] = I'_{ij}\omega'_j \tag{5.34}$$

This formula obviously holds true in the case of continuous distribution. Note also that

$$T = \frac{1}{2}\omega'_i L'_i \tag{5.35}$$

Parallel axes theorem

Consider a body-fixed frame centered at the center of mass (CM) of the rigid body. Let us denote by I'_{ij} the moments of inertial with respect to (w.r.t.) CM and by $\tilde{I}_{ij}$ the moments of inertial w.r.t. a body-fixed frame with axes parallel to the CM ones but located a distance $\vec{a}$ apart. It is easy to see that

$$\begin{aligned}\tilde{I}_{ij} &= I' = \int d^3r'\rho(r')[(\vec{r}' - \vec{a})^2\delta_{ij} - (r'-a)_i(r'-a)_j] = \int d^3r'\rho(r')({r'}^2\delta_{ij} - r'_i r'_j)\\ &- 2\delta_{ij}\vec{a}\cdot\int d^3r'\rho(r')\vec{r}' + 2\big(a_i\int d^3r'\rho(r')r'_j + i\leftrightarrow j\big) + (a^2\delta_{ij} - a_i a_j)\int d^3r'\rho(r')\\ &= I'_{ij} + M(a^2\delta_{ij} - a_i a_j)\end{aligned} \tag{5.36}$$

because by definition of the center of mass $\int d^3r'\rho(r')\vec{r}' = 0$ (and $\int d^3r'\rho(r') = M$).

Example: disc of radius R and thickness h with uniform mass distribution

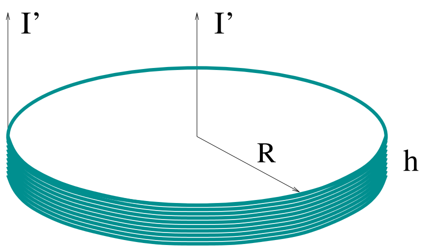

Figure 56. Disc

$$I'_{33} = \int_{-h/2}^{h/2} dz'\int_0^R ds'\, s'\int_0^{2\pi} d\phi'\Big(\frac{M}{\pi R^2 h}\Big){s'}^2 = \frac{MR^2}{2} \quad\Rightarrow\quad \tilde{I}'_{33} = I'_{33} + MR^2 = \frac{3MR^2}{2} \tag{5.37}$$

5.2.2 Principal axes

In general, the inertia tensor is not diagonal: $I'_{ij} \neq 0$ if $i \neq j$. However, because I'_{ij} is a real symmetric matrix, it can be diagonalized by an orthogonal transformation. Such transformation is of course some rotation (recall that rotations are described by orthogonal matrices). In this new rotated (body-fixed) frame

$$I'^{\text{new}} = \begin{pmatrix} I'^{\text{new}}_1 & 0 & 0 \\ 0 & I'^{\text{new}}_2 & 0 \\ 0 & 0 & I'^{\text{new}}_3 \end{pmatrix} \tag{5.38}$$

The axes forming this new frame are called principal axes. To determine the principal axes ons should use the simmetry of the body (if it has one). For example, the matrix for the disc in Fig. 62 is

$$I' = \begin{pmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix} \tag{5.39}$$

For the angular momentum we get then

$$L'^{new}_1 = I'^{new}_1 \omega'^{\text{new}}_1, \qquad L'^{new}_2 = I'^{new}_2 \omega'^{\text{new}}_2, \qquad L'^{new}_3 = I'^{new}_3 \omega'^{\text{new}}_3 \tag{5.40}$$

and for the kinetic energy

$$T = \frac{1}{2}\sum L'^{new}_i \omega'^{new}_i = \frac{1}{2}\sum I'^{new}_i (\omega'^{new}_i)^2 \tag{5.41}$$

In what follows we will always assume that we work in the principal-axes frame and omit the label "new".

5.3 Euler's equations

The motion of a rigid body is governed by equations (1.25) and (1.36)

$$M\ddot{\vec{R}} = \sum_n \vec{F}^{\text{ext}}_n, \qquad \dot{\vec{L}} = \sum_n \vec{\tau}^{\text{ext}}_n = \sum_n \vec{r}'_n \times \vec{F}^{\text{ext}}_n \equiv \vec{\Gamma}^{\text{ext}} \tag{5.42}$$

The acceleration of the center of mass (position denoted by R) is due to the sum of all *external* forces acting on the rigid body. The rate of change of the angular momentum$\vec{L}'$ relative to the CM position os due to all *external* torques calculated with the respect to the origin in CM.

From Eq. (2.20) we get

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}^{\text{ext}} \tag{5.43}$$

so

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{body}} \equiv \left(\frac{dL'_j}{dt}\right)_{\text{body}} \hat{e}'_j = \vec{\Gamma}^{\text{ext}} - \vec{\omega} \times \vec{L} \tag{5.44}$$

Multiplying this by $\hat{e}'_i$ we get

$$\left(\frac{d\vec{L}'_i}{dt}\right)_{\text{body}} = \vec{\Gamma}^{\text{ext}} \cdot \hat{e}'_i - \epsilon_{ijk}\omega'_j L'_k \tag{5.45}$$

Let us select the body-fixed frame made of principal axes so that $L'_i = I'_i\omega'_i$, then

$$I'_i\frac{d\omega'_i}{dt} = \Gamma^{\text{ext}} \cdot \hat{e}'_i - \sum_{j,k=1}^{3} \epsilon_{ijk}\omega'_j I'_k \omega'_k \tag{5.46}$$

or, in the explicit form

$$\begin{aligned} I'_1\tfrac{d\omega'_1}{dt} &= \Gamma^{\text{ext}} \cdot \hat{e}'_1 + \omega'_2\omega'_3(I'_2 - I'_3) \\ I'_2\tfrac{d\omega'_2}{dt} &= \Gamma^{\text{ext}} \cdot \hat{e}'_2 + \omega'_1\omega'_3(I'_3 - I'_1) \\ I'_1\tfrac{d\omega'_3}{dt} &= \Gamma^{\text{ext}} \cdot \hat{e}'_3 + \omega'_1\omega'_2(I'_1 - I'_2) \end{aligned} \tag{5.47}$$

Euler equations are not very simple because external torques are projected on time-dependent body-fixed principal axes. However, they are very useful for the description of torque-free motion.

5.4 Torque-free motion

In this case $\Gamma^{\text{ext}} = 0$ so in the external system $\vec{L}$ is constant and its components as seen by an observer in the external system do not change with time. However, the observer in the body-fixed frame will see the components L'_i of $\vec{L} = L'_i\hat{e}'_i$ change with time.

Euler equations for the torque-free motion

$$\begin{aligned} I'_x\tfrac{d\omega'_x}{dt} &= \omega'_y\omega'_z(I'_y - I'_z) \\ I'_y\tfrac{d\omega'_y}{dt} &= \omega'_x\omega'_z(I'_z - I'_x) \\ I'_z\tfrac{d\omega'_z}{dt} &= \omega'_x\omega'_y(I'_x - I'_y) \end{aligned} \tag{5.48}$$

Three cases

- Spherical top: $I'_x = I'_y = I'_z$. From Euler's equations (5.48) we see that ω=const (for example, rotating sphere in a free fall).
- Symmetric top: $I'_x = I'_y \neq I'_z$. From the third of Euler's equations (5.48) we see that ω'_z=const.
- Completely asymmetric top: $I'_x \neq I'_y \neq I'_z$. Analysis is complex

5.4.1 Symmetric top

As we saw, ω_z=const so the other two Euler's equations read

$$\begin{aligned} \tfrac{d\omega'_x}{dt} &= -\Omega\omega'_y \\ \tfrac{d\omega'_y}{dt} &= \Omega\omega'_x \end{aligned} \tag{5.49}$$

where $\Omega = \omega'_z \frac{I'_z - I'_x}{I'_x}$. They are easily solved by going to complex

$$\eta(t) \equiv \omega'_x(t) + i\omega'_y(t) \tag{5.50}$$

The equation (5.49) turns to

$$\dot{\eta}(t) = i\Omega\eta(t) \quad \Rightarrow \quad \eta(t) = \eta(0)e^{i\Omega t} = \big(\omega'_x(0) + i\omega'_y(0)\big)e^{i\Omega t} \tag{5.51}$$

Taking real and imaginary parts we get

$$\begin{aligned} \omega'_x(t) &= \Re\eta(t) = \omega'_x(0)\cos\Omega t - \omega'_y(0)\sin\Omega t \\ \omega'_y(t) &= \Im\eta(t) = \omega'_x(0)\sin\Omega t + \omega'_y(0)\cos\Omega t \end{aligned} \tag{5.52}$$

To visualize this motion, consider a particular set of initial conditions

$$\left.\begin{aligned} \omega'_x\big|_{t=0} &= \omega\sin\lambda \\ \omega'_y\big|_{t=0} &= 0 \\ \omega'_z\big|_{t=0} &= \omega\cos\lambda \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} \omega'_x(t) &= \omega\sin\lambda\cos\Omega t \\ \omega'_y(t) &= \omega\sin\lambda\sin\Omega t \\ \omega'_z &= \omega\cos\lambda \end{aligned}\right. \tag{5.53}$$

The solution (5.53) means that $\vec{\omega}$ (as seen in x', y', z' frame) precesses around z axis with angular velocity Ω in positive or negative direction depending on the sign of Ω, see Fig 57.

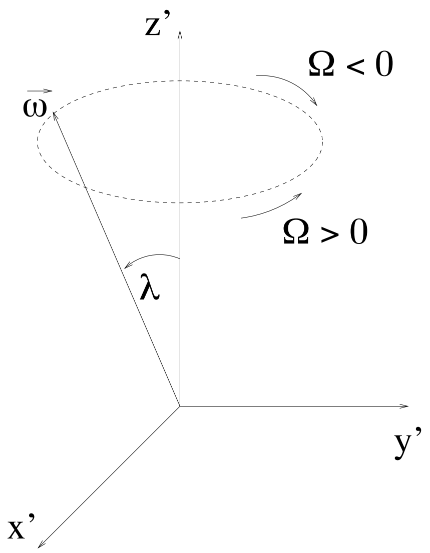

Figure 57. Symmetric top

Note that $|\vec{\omega}|$ is constant.

5.4.2 Asymmetric top

The analysis in this case is rather complex, but it is simplified by an observation that there are two constants of motion: kinetic energy and square of angular momentum

$$\begin{aligned} \vec{L}^2 &= I'^2_x\omega'^2_x + I'^2_y\omega'^2_y + I'^2_z\omega'^2_z \\ T &= \frac{1}{2}(I'_x\omega'^2_x + I'_y\omega'^2_y + I'_z\omega'^2_z) = \frac{1}{2}\vec{L}\cdot\vec{\omega} \end{aligned} \tag{5.54}$$

Let us prove the first equation. From Euler's equations (5.48) we get

$$\frac{1}{2}\frac{d}{dt}L^2 \;=\; \frac{1}{2}\frac{d}{dt}\left(I'^2_x\omega'^2_x + I'^2_y\omega'^2_y + I'^2_y\omega'^2_y\right) \;=\; I'^2_x\dot{\omega}'_x\omega'_x + I'^2_y\dot{\omega}'_y\omega'_y + I'^2_z\dot{\omega}'_z\omega'_z$$
$$= \;\omega'_y\omega'_z(I'_y - I'_z)I'_x\omega'_x + \omega'_x\omega'_z(I'_z - I'_x)I'_y\omega'_y + \omega'_x\omega'_y(I'_x - I'_y)I'_z\omega'_z \;=\; 0 \qquad (5.55)$$

Similarly, we get conservation of the kinetic energy:

$$\frac{d}{dt}T \;=\; I'_x\dot{\omega}'_x\omega'_x + I'_y\dot{\omega}'_y\omega'_y + I'_z\dot{\omega}'_z\omega'_z$$
$$= \;\omega'_y\omega'_z(I'_y - I'_z)\omega'_x + \omega'_x\omega'_z(I'_z - I'_x)\omega'_y + \omega'_x\omega'_y(I'_x - I'_y)\omega'_z \;=\; 0 \qquad (5.56)$$

One can use Eq. (5.54) to eliminate ω'_x and ω'_y from Euler's equations in favor of T, L^2, and ω'_z, then

$$\omega'_x \;=\; \omega'_x(\omega'_z, T, L^2), \qquad \omega'_y \;=\; \omega'_y(\omega'_z, T, L^2) \qquad (5.57)$$

From the third of equations (5.48) one obtains

$$t - t_0 \;=\; \frac{I'_z}{I'_x - I'_y}\int d\omega'_z \frac{1}{\omega'_x(\omega'_z, T, L^2)\omega'_y(\omega'_z, T, L^2)} \qquad (5.58)$$

The integral can be expressed in terms of elliptic integrals (see more advanced textbooks).

Part XVI

5.4.3 Motion in external (inertial) system

Angular momentum is conserved $\vec{L} \;=\;$ const.

Spherical top: $I'_x = I'_y = I'_z \Rightarrow \vec{L} = I'_x\vec{\omega} \;\;\Rightarrow\;\; \vec{\omega}$=const. The torque free motion for a spherical top reduces to a rotation about a fixed axis with angular velocity of magnitude $\omega = \frac{L}{I'_x}$

Symmetric top ($I'_x = I'_y = I_1 \neq I'_z = I_3$):

The Lagrangian $L = T$ is simplified to

$$L \;=\; \frac{I_1}{2}(\omega'^2_x + \omega'^2_y) + \frac{I_3}{2}\omega'^2_z \;=\; \frac{I_1}{2}(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) + \frac{I_3}{2}(\dot{\alpha}\cos\beta + \dot{\gamma})^2 \qquad (5.59)$$

where we used Eq. (5.25) for ω''s in the body-fixed frame. Note that the Lagrangian (5.66) does not depend on α and γ so the corresponding generalized momenta are conserved

$$p_\alpha \;\equiv\; \frac{\partial L}{\partial\dot{\alpha}} \;=\; I_1\dot{\alpha}\sin^2\beta + I_3(\dot{\alpha}\cos\beta + \dot{\gamma})\cos\beta \;=\; \text{const}$$
$$p_\gamma \;\equiv\; \frac{\partial L}{\partial\dot{\gamma}} \;=\; I_3(\dot{\alpha}\cos\beta + \dot{\gamma}) \;=\; \text{const} \qquad (5.60)$$

Let us demonstrate that p_α is the projection of angular momentum on $e_z^{(0)}$ axis in the inertial frame and p_γ on $\hat{e}'_z$ axis in the body-fixed frame. First, from Eq. (5.25) we immediately see that

$$\vec{L}\cdot\hat{e}'_z \;\equiv\; L'_z \;=\; I_3\omega'_z \;=\; I_3(\dot{\alpha}\cos\beta + \dot{\gamma}) \;=\; p_\gamma \qquad (5.61)$$

Second, from Eq. (5.34) and Eq. (5.22) we get

$$\begin{aligned}\vec{L}\cdot\hat{e}_z^{(0)} \;&=\; (I_1\omega'_x\hat{e}'_x + I_1\omega'_y\hat{e}'_y + I_3\omega'_z\hat{e}'_z)\cdot\hat{e}_z^{(0)} \;=\; I_1\omega_z + (I_3 - I_1)\omega'_z\hat{e}'_z\cdot\hat{e}_z^{(0)}\\ &=\; I_1(\dot\alpha + \dot\gamma\cos\beta) + (I_3 - I_1)(\dot\alpha\cos\beta + \dot\gamma)\cos\beta \;=\; p_\alpha \end{aligned} \tag{5.62}$$

Last, the generalized momentum p_β is a projection of $\vec{L}$ on the line of nodes $\hat{e}_\beta \;=\; \hat{e}_1^{(0)}\cos\alpha + \hat{e}_2^{(0)}\sin\alpha \;=\; \hat{e}'_1\sin\gamma + \hat{e}'_2\cos\gamma$

$$\begin{aligned} p_\beta \;&=\; \frac{\partial L}{\partial\dot\beta} \;=\; I_1\dot\beta \\ \vec{L}\cdot\hat{e}_\beta \;&=\; (I_1\omega'_x\hat{e}'_x + I_1\omega'_y\hat{e}'_y + I_3\omega'_z\hat{e}'_z)\cdot\hat{e}_\beta \;=\; I_1\vec\omega\cdot\hat{e}_\beta + (I_3 - I_1)\omega'_z\hat{e}'_z\cdot\hat{e}_\beta \;=\; I_1\dot\beta \end{aligned} \tag{5.63}$$

Now let us choose external (inertial) frame such that the (conserved) vector $\vec{L}$ points in the z direction, than since line of nodes is always orthogonal to $\hat{e}_z^{(0)}$ we get $p_\beta \;=\; \vec{L}\cdot\hat{e}_\beta \;=\; 0$.

Thus, we get ($\omega'_3 \;=\; \dot\alpha\cos\beta + \dot\gamma$)

$$\begin{aligned} p_\alpha \;&\equiv\; \frac{\partial L}{\partial\dot\alpha} \;=\; I_1\dot\alpha\sin^2\beta + I_3(\dot\alpha\cos\beta + \dot\gamma)\cos\beta \;=\; \text{const}\\ p_\gamma \;&\equiv\; \frac{\partial L}{\partial\dot\gamma} \;=\; I_3(\dot\alpha\cos\beta + \dot\gamma) \;=\; \text{const}\\ p_\beta \;&\equiv\; \frac{\partial L}{\partial\dot\beta} \;=\; 0 \qquad\Rightarrow\qquad \beta \;=\; \text{const}\\ 0 \;&=\; \frac{\partial L}{\partial\beta} \;=\; I_1\dot\alpha^2\sin\beta\cos\beta - I_3\dot\alpha\sin\beta(\dot\alpha\cos\beta + \dot\gamma) \;=\; \dot\alpha\sin\beta(I_1\dot\alpha\cos\beta - I_3\omega'_3) \end{aligned} \tag{5.64}$$

and therefore

$$\begin{aligned} \dot\alpha\cos\beta \;&=\; \frac{I_3}{I_1}\omega'_3 \;=\; \frac{I_3}{I_1}(\dot\alpha\cos\beta + \dot\gamma)\\ \Rightarrow\;\dot\gamma \;&=\; \frac{I_1 - I_3}{I_1}\omega'_3 \;=\; -\Omega\\ \omega \;&=?\; \dot\alpha\hat{e}_3^{(0)} + \dot\gamma\hat{e}'_3 \qquad\Rightarrow\qquad \vec\omega,\ \vec{L},\ \text{and } \hat{e}'_3 \text{ lie in one plane}\\ 0 \;&=\; \frac{\partial L}{\partial\beta} \;=\; I_1\dot\alpha^2\sin\beta\cos\beta - I_3\dot\alpha\sin\beta(\dot\alpha\cos\beta + \dot\gamma) \;=\; \dot\alpha\sin\beta(I_1\dot\alpha\cos\beta - I_3\omega'_3) \end{aligned} \tag{5.65}$$

Motion at $I_3 > I_1$ (see Fig. 58a):

- $\dot\alpha = \text{const}$, $\beta = \text{const}$ $\Rightarrow$ constant precession of the symmetry axis about $\vec{L}$ at a fixed polar angle β.
- $\dot\gamma = \text{const}<0$ $\Rightarrow$ constant rotation of the object about the symmetry axis. Inertial observer sees a backward motion about $\hat{e}_3^{(0)}$ while body-fixed observer sees a positive precession of $\vec\omega$ about the symmetry axis.

Motion at $I_1 > I_3$ (see Fig. 58b):

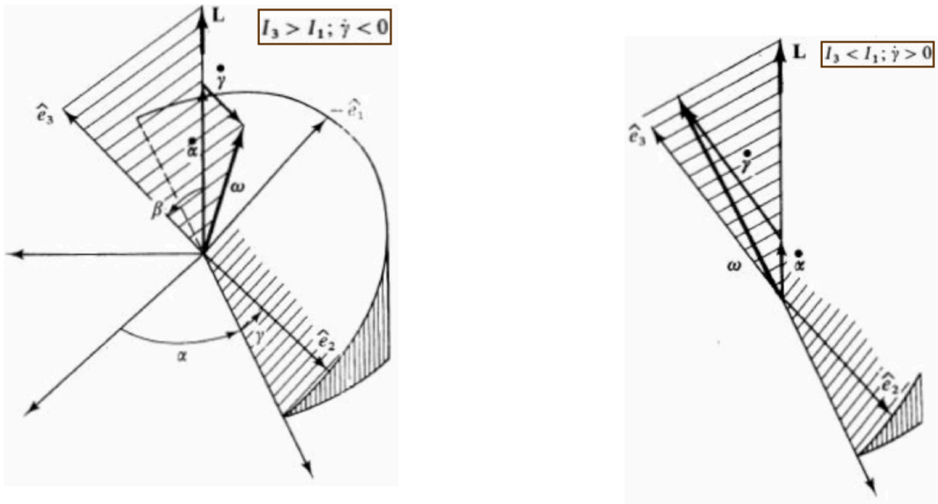

Figure 58. Symmetric top

- $\dot\alpha$ = const, β = const $\Rightarrow$ constant precession of the symmetry axis about $\vec{L}$ at a fixed polar angle β.
- $\dot\gamma$ = const>0 $\Rightarrow$ constant rotation of the object about the symmetry axis. Inertial observer sees a forward motion about $\hat{e}_3^{(0)}$ while body-fixed observer sees a positive precession of $\vec{\omega}$ about the symmetry axis.

5.5 Symmetric top with a fixed point in the gravitational field

In this case, gravity exerts a torque and changes angular momentum. From Fig. 59 we see that the direction of the torque is along the line of nodes, i.e. orthogonal to both z and z'.

The Lagrangian is now $L = T - V$

$$L = \frac{I_1}{2}(\omega'^2_x + \omega'^2_y) + \frac{I_3}{2}\omega'^2_z = \frac{I_1}{2}(\dot\alpha^2 \sin^2\beta + \dot\beta^2) + \frac{I_3}{2}(\dot\alpha\cos\beta + \dot\gamma)^2 - Mgl\cos\beta \quad (5.66)$$

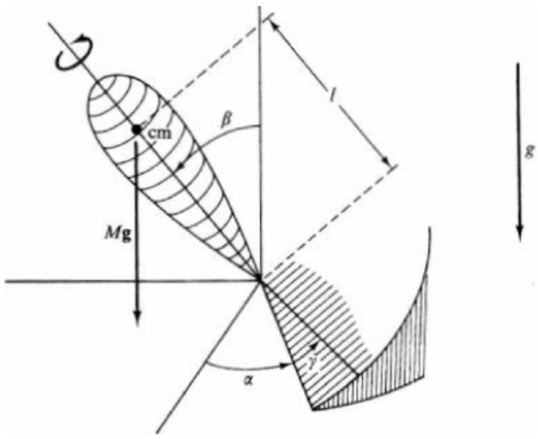

Figure 59. Symmetric top in a gravitational field

$$\begin{aligned}
p_\alpha &\equiv \frac{\partial L}{\partial\dot\alpha} = I_1\dot\alpha\sin^2\beta + I_3(\dot\alpha\cos\beta + \dot\gamma)\cos\beta = \text{const}\\
p_\gamma &\equiv \frac{\partial L}{\partial\dot\gamma} = I_3(\dot\alpha\cos\beta + \dot\gamma) = \text{const}\\
p_\beta &\equiv \frac{\partial L}{\partial\dot\beta} \neq 0 \qquad \Rightarrow \qquad \beta \neq \text{const} \qquad (5.67)\\
I_1\ddot\beta &= \frac{\partial L}{\partial\beta} = I_1\dot\alpha^2\sin\beta\cos\beta - I_3\dot\alpha\sin\beta(\dot\alpha\cos\beta + \dot\gamma) + Mgl\sin\beta
\end{aligned}$$

From the first two equations

$$\dot\alpha = \frac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}, \qquad \dot\gamma = p_\gamma\Big(\frac{1}{I_3} + \frac{\cos^2\beta}{I_1\sin^2\beta}\Big) - \frac{p_\alpha\cos\beta}{I_1\sin^2\beta} \quad (5.68)$$

and therefore the Euler-Lagrange equation for β turns to

$$\begin{aligned}
I_1\ddot\beta &= \frac{\partial L}{\partial\beta} \quad\Rightarrow\quad I_1\dot\alpha^2\sin\beta\cos\beta - I_3\dot\alpha\sin\beta(\dot\alpha\cos\beta + \dot\gamma) + Mgl\sin\beta\\
\Rightarrow I_1\ddot\beta &= \frac{\cos\beta}{I_1\sin^3\beta}(p_\alpha^2 - 2p_\alpha p_\gamma\cos\beta + p_\gamma^2) - \frac{p_\alpha p_\gamma}{I_1\sin\beta} + Mgl\sin\beta \qquad (5.69)
\end{aligned}$$

If we solve this equation, we can find α and γ from Eqs. (5.68).

5.5.1 Method of effective potential

First integral of Eq. (5.71)

$$
\begin{aligned}
I_1\dot\beta\ddot\beta &= \dot\beta\Big(\frac{\cos\beta}{I_1\sin^3\beta}(p_\alpha^2 - 2p_\alpha p_\gamma\cos\beta + p_\gamma^2) - \frac{p_\alpha p_\gamma}{I_1\sin\beta} + Mgl\sin\beta\Big) \\
\Rightarrow\ \frac{d}{dt}\frac{I_1\dot\beta^2}{2} &= -\frac{d}{dt}\Big(\frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta\Big) \\
\Rightarrow\ E &= \frac{1}{2}I_1\dot\beta^2 + V_{\text{eff}}(\beta) = \text{const} \qquad (5.70)
\end{aligned}
$$

$\Rightarrow$ we get conservation of energy

$$
\begin{aligned}
E &= \frac{1}{2}I_1\dot\beta^2 + V_{\text{eff}}(\beta) = \text{const} \\
V_{\text{eff}}(\beta) &= \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta \qquad (5.71)
\end{aligned}
$$

This is the one-dimensional problem with effective potential $V_{\rm eff}(\beta)$ shown in Fig. 60. Potential diverges as $\beta \to 0, \pi$ which corresponds to the "angular momentum barrier". There

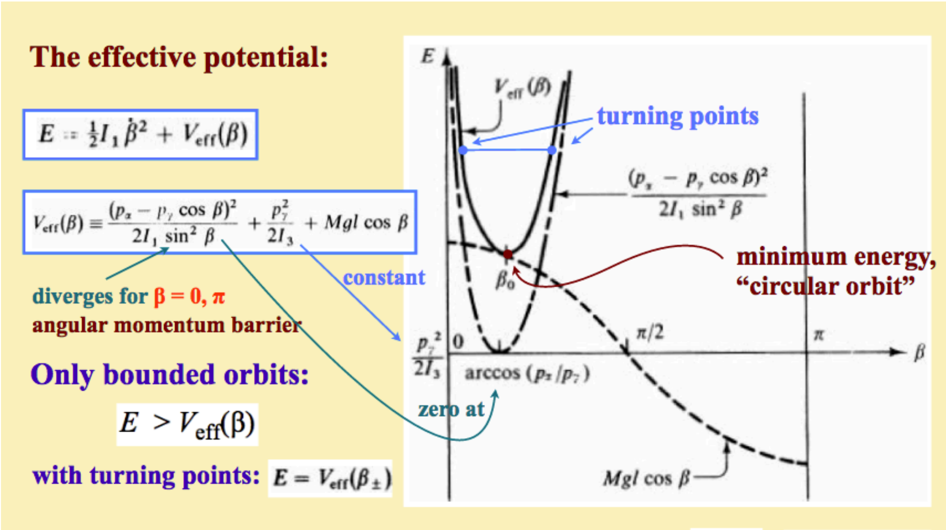

Figure 60. Effective potential

are only bounded orbits with turning points $E = V_{\rm eff}(\beta_\pm)$. As $E = V_{\rm eff}^{\rm min}$ the orbit becomes circular.

Let us study the circular orbit. The corresponding β_0 is found from the equation

$$\frac{\partial V_{\rm eff}}{\partial \beta}\Big|_{\beta=\beta_0} = 0 \quad \Rightarrow \quad \frac{\cos\beta}{I_1 \sin^3\beta}(p_\alpha^2 - 2p_\alpha p_\gamma \cos\beta + p_\gamma^2) - \frac{p_\alpha p_\gamma}{I_1 \sin\beta} + Mgl\sin\beta = 0 \quad (5.72)$$

Once we know β_0, from Eq. (5.68) we see that

$$\alpha = \frac{p_\alpha - p_\gamma \cos\beta_0}{I_1 \sin^2\beta_0} t, \qquad \gamma = p_\gamma\Big(\frac{1}{I_3} + \frac{\cos^2\beta_0}{I_1 \sin^2\beta_0}\Big)t - \frac{p_\alpha \cos\beta_0}{I_1 \sin^2\beta_0} t \quad (5.73)$$

Let us now study small oscillations about steady motion. Expanding

$$\beta(t) = \beta_0 + \eta(t)$$

we get

$$V_{\rm eff}(\beta(t)) = V_{\rm eff}(\beta_0) + \eta(t)\frac{\partial V_{\rm eff}(\beta)}{\partial \beta}\Big|_{\beta=\beta_0} + \frac{1}{2}\eta^2(t)\frac{\partial^2 V_{\rm eff}(\beta)}{\partial^2 \beta}\Big|_{\beta=\beta_0} + O(\eta^3) \quad (5.74)$$

Since steady motion corresponds to β_0 such that $\frac{\partial V_{\rm eff}(\beta)}{\partial \beta}\Big|_{\beta=\beta_0} \;=\; 0$, we get

$$V_{\rm eff}(\beta(t)) \;=\; V_{\rm eff}(\beta_0) + \frac{1}{2}\eta^2(t) I_1 \Omega^2 \tag{5.75}$$

where

$$\Omega^2 \;=\; \frac{1}{I_1}\frac{\partial^2 V_{\rm eff}(\beta)}{\partial^2 \beta}\Big|_{\beta=\beta_0} \;=\; \frac{p_\alpha p_\gamma - I_1 M g l (3 - 4\sin^2\beta_0)}{I_1^2 \cos\beta_0} \tag{5.76}$$

The energy (5.71) takes the form

$$E \;=\; \frac{1}{2} I_1\big(\dot\eta^2 + \Omega^2\eta^2\big) + V_{\rm eff}(\beta_0) \;=\; \text{const} \quad \Rightarrow \quad \dot\eta^2 + \Omega^2 \eta^2 \;=\; \text{const} \tag{5.77}$$

This is the harmonic oscillator problem with the solution

$$\beta(t) \;=\; \beta_0 + \eta_0 \cos(\Omega t + \phi_0) \tag{5.78}$$

which describes simple oscillations about β_0. They are stable if $\Omega^2 > 0$. Looking back at $p_\alpha \;=\; \vec{L}\cdot\hat{e}_z^{(0)}$ and $p_\gamma \;=\; \vec{L}\cdot\hat{e}'_z$ we see that $\Omega^2 > 0$ requires sufficiently large angular momentum.

In the case of "sleeping top" (= rapidly spinning top with vertical symmetry axis) we have $\Leftrightarrow \;\; \cos\beta_0 \;=\; 1$, $\hat{e}_3^{(0)} = \hat{e}'_3$, $p_\gamma = p_\alpha$ so the stability condition $\Omega^2 > 0$ reads

$$p_\gamma^2 \;>\; 4 I_1 M g l \tag{5.79}$$

5.5.2 Precession and nutation

Small nutation around steady trajectory are described by Eqs. (5.75) and (5.73)

$$\begin{aligned} \beta(t) &= \beta_0 + \eta_0 \cos(\Omega t + \phi_0) \\ \dot{\alpha}(t) &= \frac{p_\alpha - p_\gamma \cos\beta(t)}{I_1 \sin^2 \beta(t)} \simeq \dot{\alpha}_0 + \eta(t)\dot{\alpha}_1 \\ \dot{\gamma} &= p_\gamma \Big(\frac{1}{I_3} + \frac{\cos^2 \beta_0}{I_1 \sin^2 \beta_0} - \frac{p_\alpha \cos\beta_0}{I_1 \sin^2 \beta_0} \Big) \simeq \dot{\gamma}_0 + \eta(t)\dot{\gamma}_1 \end{aligned} \tag{5.80}$$

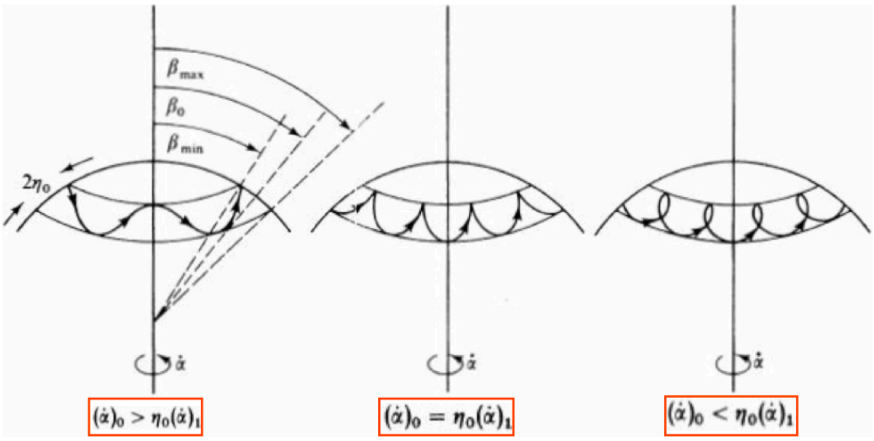

Figure 61. Nutation of a symmetric top

Part XVII

6 Hamiltonian dynamics

6.1 Hamilton's equations

In the Hamiltonian formulation, generalized coordinates and generalized momenta appear on equal footing. We write down generalized momenta as usually

$$p_\alpha \equiv \frac{\partial L}{\partial q_\alpha} \tag{6.1}$$

define

$$H = \sum_\alpha p_\alpha \dot{q}_\alpha - L(\{q, \dot{q}\}, t) \tag{6.2}$$

and re-express all $\dot{q}_\alpha$ in Eq. (6.2) in terms of p_α and q_α and maybe t. The obtained function is called Hamiltonian

$$H(p_\alpha, q_\alpha, t) = \sum_\alpha p_\alpha \dot{q}_\alpha(\{q, p\}, t) - L\big(\{q_\alpha, (\dot{q}(\{q, p\}, t))\}, t\big) \tag{6.3}$$

Hereafter $\{q,\dot{q}\}$ denotes a set of $q_\alpha, \dot{q}_\alpha$, $\{q,\dot{q}\}$ a set of q_α, p_α etc.

In this formulation, the Euler-Lagrange equations are expressed as Hamilton's equations

$$\begin{aligned}\frac{dq_\alpha}{dt} &= \frac{\partial H(\{q,p\},t)}{\partial p_\alpha}\\ \frac{dp_\alpha}{dt} &= -\frac{\partial H(\{q,p\},t)}{\partial q_\alpha}\end{aligned} \tag{6.4}$$

Let us prove that they follow from Lagrange equations

$$\begin{aligned}\frac{\partial H(\{q,p\},t)}{\partial p_\alpha} &= \dot{q}_\alpha + \sum_\alpha \frac{\partial \dot{q}_\alpha(\{q,p\},t)}{dp_\alpha}\Big(p_\alpha - \frac{\partial L(\{q,\dot{q}\},t)}{\partial \dot{q}_\alpha}\Big) = \dot{q}_\alpha\\ \frac{\partial H(\{q,p\},t)}{\partial q_\alpha} &= \sum_\beta \frac{\partial \dot{q}_\beta(\{q,p\},t)}{\partial q_\alpha} p_\beta - \frac{\partial L(\{q,\dot{q}(\{q,p\},t)\},t)}{\partial q_\alpha}\end{aligned} \tag{6.5}$$

$$= \sum_\beta \frac{\partial \dot{q}_\beta(\{q,p\},t)}{\partial q_\alpha} p_\beta - \frac{\partial L(\{q,\dot{q}\},t)}{\partial q_\alpha}\Big|_{\dot{q}_\beta=\dot{q}_\beta(\{q,p\},t)} - \sum_\beta \frac{\partial L(\{q,\dot{q}\},t)}{\partial \dot{q}_\beta}\Big|_{\dot{q}_\beta=\dot{q}_\beta(\{q,p\},t)} \frac{\partial \dot{q}_\beta(\{q,p\},t)}{\partial q_\alpha}$$

$$= -\frac{\partial L(\{q,\dot{q}\},t)}{\partial q_\alpha}\Big|_{\dot{q}_\beta=\dot{q}_\beta(\{q,p\},t)} = -\frac{d}{dt}\frac{\partial L(\{q,\dot{q}\},t)}{\partial \dot{q}_\alpha}\Big|_{\dot{q}_\beta=\dot{q}_\beta(\{q,p\},t)} = -\frac{d}{dt}p_\alpha$$

Mathematically, the Lagrange equations are n differential equations of the second order while Hamilton equations are $2n$ first-order differential equations (equivalent to n second-order equations).

Note that if the Lagrangian does not depend on time explicitly ($\equiv \frac{\partial L}{\partial t} = 0$), the Hamiltonian (6.3) is a constant of motion:

$$\frac{dH}{dt} = \sum_\alpha \Big(\frac{\partial H}{\partial p_\alpha}\frac{dp_\alpha}{dt} + \frac{\partial H}{\partial q_\alpha}\frac{dq_\alpha}{dt}\Big) = \sum_\alpha \Big(-\frac{\partial H}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha} + \frac{\partial H}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha}\Big) = 0 \tag{6.6}$$

Moreover, if potential energy and the (holonomic) constraints do not depend on time, the conserved Hamiltonian is the energy, see Eq. (3.111).

Example: free particle in one dimension

$$L = \frac{m\dot{x}^2}{2} \quad\Rightarrow\quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad\Rightarrow\quad H = p\dot{x} - \frac{m\dot{x}^2}{2} = \frac{p_x^2}{2m} \tag{6.7}$$

Equations of motion (6.4):

$$\frac{dp_x}{dt} = \frac{\partial H}{\partial x} = 0 \quad\Rightarrow\quad p = \text{const}, \qquad \frac{dx}{dt} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad\Rightarrow\quad x = x_0 + \frac{p_x}{m}t \tag{6.8}$$

It is possible to obtain Hamilton's equations (6.4) from variational principle

$$\delta\Big(\int_{t_1}^{t_2} dt\,\Big[\sum_\alpha \dot{q}_\alpha(\{q,p\},t)p_\alpha - H(\{q,p\},t)\Big]\Big) = 0 \tag{6.9}$$

with boundary conditions fixed for both q's and p's:

$$\delta q(t_1) = \delta q(t_2) = 0 \quad\text{and}\quad \delta p(t_1) = \delta p(t_2) = 0 \tag{6.10}$$

The action integral (3.83), expressed in terms of H, is stationary with respect to independent variation of q's and p's (for proof see the textbook).

6.2 Example: charged particle in the electromagnetic field

6.2.1 Lagrangian

The motion of charged particle in the electromagnetic field is governed by Newton's laws with Lorentz force

$$m\ddot{\vec{r}} \;=\; e\big(\vec{E} + \frac{\dot{\vec{r}}}{c} \times \vec{B}\big) \tag{6.11}$$

(we use cgs system here). Let us prove that the this equation of motion can be obtained from the Lagrangian

$$L(\vec{r}, \dot{\vec{r}}, t) \;=\; \frac{1}{2} m\dot{\vec{r}}^2 - e\Phi(\vec{r}, t) + e\frac{\dot{\vec{r}}}{c} \cdot \vec{A}(\vec{r}, t) \tag{6.12}$$

where $\Phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ are scalar and vector potentials defined by

$$\begin{aligned} \vec{E}(\vec{r}, t) \;&=\; -\nabla\Phi(\vec{r}, t) - \frac{1}{c}\frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \\ \vec{B}(\vec{r}, t) \;&=\; \nabla \times \vec{A}(\vec{r}, t) \end{aligned} \tag{6.13}$$

The partial derivatives of the Lagrangian (6.12) are

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}_i} \;&=\; m\dot{r}_i + \frac{e}{c} A_i(\vec{r}, t) \\ \frac{\partial L}{\partial r_i} \;&=\; -e\partial_i \Phi + e\frac{\dot{r}_j}{c}\frac{\partial A_j(\vec{r}, t)}{\partial r_i} \end{aligned} \tag{6.14}$$

so Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\big(m\dot{r}_i + \frac{1}{c} A_i(\vec{r}, t)\big) \;&=\; -e\partial_i\Phi + \frac{1}{c}\frac{\partial A_i(\vec{r}, t)}{\partial t} + \frac{\dot{r}_j}{c}\frac{\partial A_j(\vec{r}, t)}{\partial r_i} \\ \Rightarrow\; m\ddot{r}_i \;&=\; -\frac{1}{c}\frac{\partial A_i(\vec{r}, t)}{\partial r_j}\dot{r}_j - e\partial_i\Phi - \frac{1}{c}\frac{\partial A_i(\vec{r}, t)}{\partial t} + e\frac{\dot{r}_j}{c}\frac{\partial A_j(\vec{r}, t)}{\partial r_i} \end{aligned} \tag{6.15}$$

Let us now rewrite the Lorentz force (6.11) in terms of potentials (6.13)

$$\begin{aligned} m\ddot{r}_i \;&=\; e\big(\vec{E}_i + \frac{1}{c}\epsilon_{ijk}\dot{r}_j\vec{B}_k\big) \;=\; -e\partial_i\Phi(\vec{r}, t) - \frac{1}{c}\frac{\partial A_i(\vec{r}, t)}{\partial t} + \frac{e}{c}\epsilon_{ijk}\epsilon_{klm}\dot{r}_j\partial_l A_m(r, t) \qquad (6.16) \\ &=\; -e\partial_i\Phi(\vec{r}, t) - \frac{1}{c}\frac{\partial A_i(\vec{r}, t)}{\partial t} + \frac{e}{c}\big(\dot{r}_j\partial_i A_j(r, t) - \dot{r}_j\partial_j A_i(r, t)\big) \;=\; \text{r.h.s. of Eq. (6.15)} \end{aligned}$$

where we used $\epsilon_{ijk}\epsilon_{lmk} \;=\; \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ (summation over k is implied as usual). Thus, the Eq. (6.12) is the correct Lagrangian for a particle in electromagnetic field.

6.2.2 Hamiltonian

Canonical momenta:

$$p_i \;\equiv\; \frac{\partial L}{\partial \dot{r}_i} \;=\; m\dot{r}_i(t) + \frac{e}{c}A_i(\vec{r}, t) \quad\Leftrightarrow\quad \vec{p} \;=\; m\dot{\vec{r}} + \frac{e}{c}\vec{A} \tag{6.17}$$

If Φ and $\vec{A}$ do not depend on x, the corresponding generalized momentum p_x is conserved but not equal to ordinary momentum $m\dot{x}$.

The Hamiltonian is

$$H = p_i\dot{r}_i - L = m\dot{\vec{r}}\cdot\Big(\dot{\vec{r}} + \frac{e}{c}\vec{A}\Big) - \frac{1}{2}m\dot{\vec{r}}^2 + e\Phi - e\frac{\dot{\vec{r}}}{c}\cdot\vec{A} = \frac{1}{2}m\dot{\vec{r}}^2(t) + e\Phi(\vec{r},t) \quad (6.18)$$

which is clearly the sum of kinetic and potential energy of the particle in electromagnetic field. Note, however, that the Hamilton must be expressed in terms of p_i rather than $\dot{x}_i$. We get

$$H(\vec{p},\vec{r},t) = \frac{1}{2m}\Big[\vec{p} - \frac{e}{c}\vec{A}(\vec{r},t)\Big]^2 + e\Phi(\vec{r},t) \quad (6.19)$$

Let us check that corresponding Hamilton equations (6.4) reproduce Newton's 2nd law with Lorentz force.

$$\begin{aligned}\frac{dr_i}{dt} &= \frac{\partial H}{\partial p_i} = \frac{1}{m}\Big(p_i - \frac{e}{c}A_i(\vec{r},t)\Big)\\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial r_i} = -\frac{e}{mc}\frac{\partial\vec{A}}{\partial r_i}\cdot\Big(\vec{p} - \frac{e}{c}\vec{A}(\vec{r},t)\Big) - e\partial_i\Phi(\vec{r},t)\end{aligned} \quad (6.20)$$

Differentiating the first equation with respect to t we get

$$\begin{aligned}\ddot{r}_i &= \frac{1}{m}\Big(\dot{p}_i - \frac{e}{c}\frac{\partial A_i(\vec{r},t)}{\partial t} - \frac{e}{c}\frac{\partial A_i(\vec{r},t)}{\partial r_j}\dot{r}_j\Big)\\ &= -\frac{e}{mc}\frac{\partial A_i(\vec{r},t)}{\partial t} - \frac{e}{mc}\frac{\partial A_i(\vec{r},t)}{\partial r_j}\dot{r}_j - \frac{e}{mc}\frac{\partial\vec{A}}{\partial r_i}\cdot(m\dot{\vec{r}}) - e\partial_i\Phi(\vec{r},t)\\ &= eE_i(\vec{r},t) + \frac{e}{mc}\dot{r}_j[\partial_i A_j(\vec{r},t) - \partial_j A_i(\vec{r},t)] = eE_i(\vec{r},t) + \frac{e}{mc}\epsilon_{ijk}\dot{r}_j B_k(\vec{r},t)\end{aligned} \quad (6.21)$$

which is Eq. (6.11). (In the last line we used the formula from Eq. (6.16)).

6.3 Canonical transformations

6.3.1 Point transformations in the Lagrangian formulation

Suppose we have a Lagrangian $L(q_\alpha, \dot{q}_\alpha, t)$, $\alpha = 1, 2...n$. Consider so-called point transformations

$$q_\alpha = q_\alpha(Q_1, Q_2, ...Q_n, t), \qquad \alpha = 1, 2...n \quad (6.22)$$

which are assumed to be non-singular and invertible

$$Q_\alpha = Q_\alpha(q_1, q_2, ...q_n, t), \qquad \alpha = 1, 2...n \quad (6.23)$$

Theorem:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_\alpha} = \frac{\partial L}{\partial q_\alpha} \quad\Rightarrow\quad \frac{d}{dt}\frac{\partial L}{\partial\dot{Q}_\alpha} = \frac{\partial L}{\partial Q_\alpha} \quad (6.24)$$

<u>Proof:</u>
Consider

$$\begin{aligned}&\frac{\partial}{\partial\dot{Q}_\alpha}L\Big(q_\beta(Q_\gamma,t), \dot{q}_\beta(Q_\gamma,t), t\Big) = \frac{\partial}{\partial\dot{Q}_\alpha}L\Big(q_\beta(Q_\gamma,t), \frac{d}{dt}q_\beta(Q_\gamma,t), t\Big)\\ &= \frac{\partial}{\partial\dot{Q}_\alpha}L\Big(q_\beta(Q_\gamma,t), \frac{\partial}{\partial t}q_\beta(Q_\gamma,t) + \frac{\partial q_\beta}{\partial Q_\gamma}\dot{Q}_\gamma, t\Big) = \frac{\partial L}{\partial\dot{q}_\beta}\frac{\partial q_\beta}{\partial Q_\alpha}\end{aligned} \quad (6.25)$$

Taking now derivative with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}\frac{\partial}{\partial \dot{Q}_\alpha}L\Big(q_\beta(Q_\gamma,t),\dot{q}_\beta(Q_\gamma,t),t\Big) &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\beta}\frac{\partial q_\beta}{\partial Q_\alpha} = \frac{\partial q_\beta}{\partial Q_\alpha}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\beta} + \frac{\partial L}{\partial \dot{q}_\beta}\frac{d}{dt}\frac{\partial q_\beta}{\partial Q_\alpha} \\ &= \frac{\partial L}{\partial q_\beta}\frac{\partial q_\beta}{\partial Q_\alpha} + \frac{\partial L}{\partial \dot{q}_\beta}\frac{\partial \dot{q}_\beta}{\partial Q_\alpha} = \frac{\partial L}{\partial Q_\alpha} \end{aligned} \tag{6.26}$$

where we used

$$\begin{aligned}\frac{\partial}{\partial Q_\alpha}\frac{d}{dt}q_\beta(Q_\gamma,t) &= \frac{\partial}{\partial Q_\alpha}\Big(\frac{\partial}{\partial t}q_\beta(Q_\gamma,t) + \frac{\partial q_\beta}{\partial Q_\gamma}\dot{Q}_\gamma\Big) = \frac{\partial}{\partial t}\frac{\partial q_\beta}{\partial Q_\alpha} + \dot{Q}_\gamma\frac{\partial^2 q_\beta}{\partial Q_\alpha \partial Q_\gamma} \\ \frac{d}{dt}\frac{\partial}{\partial Q_\alpha}q_\beta(Q_\gamma(t),t) &= \frac{\partial^2 q_b}{\partial Q_\alpha \partial Q_\gamma}\dot{Q}_\gamma + \frac{\partial}{\partial t}\frac{\partial q_\beta}{\partial Q_\alpha} \end{aligned} \tag{6.27}$$

$\Rightarrow \frac{d}{dt}\frac{\partial q_\beta}{\partial Q_\alpha} = \frac{\partial}{\partial Q_\alpha}\frac{dq_\beta}{dt}$ (nontrivial due to $\frac{d}{dt}$ rather than $\frac{\partial}{\partial t}$). The proof is complete.

Example: transition to spherical coordinates $(x,y,z) \rightarrow (r,\theta,\phi)$ for a particle in a central potential $V(r)$.

- In Cartesian coordinates $L_{\rm C} = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2 - V\big(\sqrt{x^2+y^2+z^2}\big) \Rightarrow$ Euler-Lagrange equations are complicated
- In spherical coordinates $L_{\rm sph} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta} + r^2\dot{\phi}^2\sin^2\theta) - V(r) \Rightarrow$ Euler-Lagrange equations are simple

Part XVIII

6.3.2 Transformations in the Hamiltonian formulation

The coordinates and momenta enter the Hamiltonian formulation on equal footing so it is natural to consider transformations of the type

$$\left.\begin{aligned} q_\alpha &= q_\alpha(Q_\beta,P_\beta,t) \\ p_\alpha &= p_\alpha(Q_\beta,P_\beta,t)\end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} Q_\alpha &= Q_\alpha(q_\beta,p_\beta,t) \\ P_\alpha &= P_\alpha(q_\beta,p_\beta,t)\end{aligned}\right. \tag{6.28}$$

which are again assumed to be invertible and non-singular. Note that this class of transformations is more wide than point transformations (6.22) since, for example, the new coordinates may depend on old velocities.

Still, the transformations (6.28) should describe the same physical problem (same Euler-lagrange equations $\leftrightarrow$ same Newton's 2nd law). For a general set of transformations (6.28) this is not always the case. For example, consider a free particle with $H = \frac{p^2}{2m}$ described by Hamilton equations (6.7) and (6.8)

$$\frac{dp}{dt} = \frac{\partial H}{\partial q} = 0 \quad \Rightarrow \quad p = \text{const}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \Rightarrow \quad q = q_0 + \frac{p}{m}t \tag{6.29}$$

or Euler-Lagrange equations ($L = \frac{m\dot{x}^2}{2}$)

$$\frac{\partial L}{\partial q} = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \Rightarrow \quad q = q_0 + \frac{p}{m}t \tag{6.30}$$

Now consider a transformation

$$Q = \frac{p^2}{m^2}, \quad P = q \tag{6.31}$$

The Hamiltonian is now

$$H(P,Q) = \frac{1}{2}Qm \quad \Rightarrow \quad L(Q,\dot{Q}) = -\frac{1}{2}Qm \tag{6.32}$$

and the Euler-Lagrange equation reads

$$\frac{\partial L}{\partial Q} = -\frac{m}{2}, \quad \frac{\partial L}{\partial \dot{Q}} = 0 \quad \Rightarrow \quad m = 0 \tag{6.33}$$

which is not the correct description of our problem with $m \neq 0$.

On the other hand, consider

$$Q = -p, \quad P = q \tag{6.34}$$

The Hamiltonian is now

$$H(P,Q) = \frac{Q^2}{2m} \quad \Rightarrow \quad L(Q,\dot{Q},t) = (A+Bt)\dot{Q} - \frac{Q^2}{2m} \tag{6.35}$$

and the Euler-Lagrange equation gives

$$\frac{\partial L}{\partial Q} = -\frac{Q}{2m}, \quad \frac{\partial L}{\partial \dot{Q}} = A+Bt \quad \Rightarrow \quad B = -\frac{Q}{m} \quad \Rightarrow \quad Q = -Bm = \text{const}$$
$$P = \frac{\partial L}{\partial \dot{Q}} = A+Bt = P_0 - \frac{Q}{m}t \tag{6.36}$$

which gives Eq. (6.30) after inverse transformation $p = -Q$, $q = P$.

We see that not all of the transformations (6.28) are acceptable, only those which do not change physics of our problem $\Leftrightarrow$ do not change Euler-Lagrange equations $\Leftrightarrow$ do not change Hamilton equations. Such transformations are called *canonical transformations.*

Definition: the transformations

$$\left.\begin{array}{rcl} q_\alpha &=& q_\alpha(Q_\beta,P_\beta,t) \\ p_\alpha &=& p_\alpha(Q_\beta,P_\beta,t) \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{rcl} Q_\alpha &=& Q_\alpha(q_\beta,p_\beta,t) \\ P_\alpha &=& P_\alpha(q_\beta,p_\beta,t) \end{array}\right. \tag{6.37}$$

are called canonical if for the new Hamiltonian

$$\tilde{H}(Q,P,t) \equiv H\big(Q(\{q,p\},t), P(\{q,p\},t), t\big) \tag{6.38}$$

we have the Hamilton equations of the same form:

$$\left.\begin{array}{rcl} \frac{dq_\alpha}{dt} &=& \frac{\partial H(\{q,p\},t)}{\partial p_\alpha} \\ \frac{dp_\alpha}{dt} &=& -\frac{\partial H(\{q,p\},t)}{\partial q_\alpha} \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{rcl} \frac{dQ_\alpha}{dt} &=& \frac{\partial \tilde{H}(\{Q,P\},t)}{\partial P_\alpha} \\ \frac{dP_\alpha}{dt} &=& -\frac{\partial \tilde{H}(\{Q,P\},t)}{\partial Q_\alpha} \end{array}\right. \tag{6.39}$$

The canonical transformations may be very useful. Imagine that we have found such transformation $(q,p) \to Q,P$ that $\tilde{H} = \tilde{H}(P)$, then all momenta P are conserved so the solution of Hamilton equations in terms of Q and P is trivial.

Thus, the task is to find a suitable canonical transformation which simplifies our problem. This sometimes can be done with the method of finding the suitable *generating function.*

6.3.3 Method of generating function

Recall that one can derive the Hamilton's equations from a modified form of the least action principle applied to

$$\int_{t_1}^{t_2} \Big(\sum_i p_i \dot{q}_i - H(q,p;t) \Big)\, dt \ , \tag{6.40}$$

with constraints that both $\delta p_i(t)$ and $\delta q_i(t)$ vanish at the endpoints $t = t_1$ and $t = t_2$, see Eq. (6.9). We should have

$$\delta \int_{t_1}^{t_2} \Big(\sum_i p_i \dot{q}_i - H(q,p;t) \Big)\, dt = 0 \tag{6.41}$$

in old variables and

$$\delta \int_{t_1}^{t_2} \Big(\sum_i P_i \dot{Q}_i - \tilde{H}(Q,P;t) \Big)\, dt = 0 \tag{6.42}$$

in the new variables. These two relations are satisfied simultaneously if the two integrands differ by full time derivative of some function F:

$$\sum_i P_i \dot{Q}_i - \tilde{H}(Q,P;t) = \sum_i p_i \dot{q}_i - H(q,p;t) + \frac{dF}{dt} \ . \tag{6.43}$$

Indeed, the variation of dF/dt is

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt}\, dt = \delta(F(t_1) - F(t_2)) = \delta(\text{const}) \ , \tag{6.44}$$

where "const" is independent of the shape of $\delta q_i(t)$ and $\delta p_i(t)$, since $\delta q_i(t_1) = \delta q_i(t_2) = 0$ and $\delta p_i(t_1) = \delta p_i(t_2) = 0$.

Rewriting Eq. (6.43) as

$$dF = \sum_i P_i dQ_i - \sum_i p_i dq_i + (H - \tilde{H})\, dt \ , \tag{6.45}$$

we find that

$$p_i = -\frac{\partial F}{\partial q_i} \quad , \quad P_i = \frac{\partial F}{\partial Q_i} \quad , \quad \tilde{H} = H - \frac{\partial F}{\partial t} \ . \tag{6.46}$$

The function F is called the *generating function* of the canonical transformation.

If $F(q,p,Q,P;t)$ does not depend on t explicitly, we can write $\tilde{H} = H$ and

$$\frac{dF}{dt} = \sum_i P_i \frac{dQ_i}{dt} - \sum_i p_i \frac{dq_i}{dt} \ , \tag{6.47}$$

i.e., in this case $H(q,p)$ and $\tilde{H}(Q,P)$ are the same functions just written in different variables: $H(p,q) = \tilde{H}(Q(q,p), P(q,p))$ and

$$dF = \sum_i P_i dQ_i - \sum_i p_i dq_i \ . \tag{6.48}$$

Since the differential of F is determined by changes in (differentials of) q_i and Q_i, Eq. (6.48) generates F as a function of $2s$ variables $\{q_i, Q_i\}$.
NB: In general, since new and old variables are related by $2s$ equations

$$Q_i = Q_i(q,p,t) \quad , \quad P_i = P_i(q,p,t) \; , \tag{6.49}$$

the function $F(\{q_i,p_i\},\{Q_j,P_j\};t)$ has only $2s = 4s-2s$ canonical variables as independent. Thus, the possible choices are $F_1(q,Q;t), F_2(q,P;t), F_3(p,Q;t), F_4(p,P;t)$. A particular choice is determined by convenience of application to a particular problem.

Example: take $F = \sum_i q_i Q_i$, then

$$p_i = -\frac{\partial F}{\partial q_i} = -Q_i \quad , \quad P_i = \frac{\partial F}{\partial Q_i} = q_i \quad , \tag{6.50}$$

so this transformation interchanges coordinates and momenta.
Another example:
$F(q,Q) \;=\; -\frac{m\omega}{2} q^2 \cot Q$ for harmonic oscillator with $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$.

$$\begin{aligned} \frac{dF}{dQ} &= P = \frac{mq^2\omega}{2\sin^2 Q} \quad \Rightarrow \quad \frac{1}{\sin Q} = \frac{1}{q}\sqrt{\frac{2P}{m\omega}} \quad \Leftrightarrow \quad q = \sqrt{\frac{2P}{m\omega}} \sin Q \\ -\frac{dF}{dq} &= p = m\omega q \cot Q = \sqrt{2Pm\omega}\cos Q \end{aligned} \tag{6.51}$$

The Hamiltonian in new coordinates takes the form

$$\tilde{H}(Q,P) \;=\; H\big(q(Q,P),p(Q,P)\big) \;=\; \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \;=\; \omega P \tag{6.52}$$

Thus, in new variables the Hamiltonian is cyclic in Q so

$$\begin{aligned} \frac{\partial \tilde{H}(Q,P)}{\partial Q} &= -\dot{P} = 0 \quad \Rightarrow \quad P = P_0 = \text{const} \\ \frac{\partial \tilde{H}(Q,P)}{\partial P} &= \dot{Q} = \omega \quad \Rightarrow \quad Q = Q_0 + \omega t \end{aligned} \tag{6.53}$$

and the solution in terms of the original coordinates

$$\begin{aligned} q &= \sqrt{\frac{2P_0}{m\omega}} \sin(Q_0 + \omega t) \\ p &= \sqrt{2P_0 m\omega}\cos(Q_0+\omega t) \end{aligned} \tag{6.54}$$

takes the familiar form $q = A_0\cos(\omega t + \phi_0)$ with $\phi_0 = Q_0$ and $A_0 = \sqrt{2P_0 m\omega}$.

In general, switching from a generating function $F(q,Q;t)$ that depends on the variables q, Q to another generating function that depends, say, on P and q is accomplished by the *Legendre transformation*

$$\begin{aligned} -d\Phi \equiv d\big(F - \sum_i P_i Q_i\big) & \\ &= \Big[\sum_i P_i dQ_i - \sum_i p_i dq_i + (H-\tilde{H})dt\Big] - \sum_i (P_i dQ_i + Q_i dP_i) \\ &= -\sum_i Q_i dP_i - \sum_i p_i dq_i + (H-\tilde{H})dt \; . \end{aligned} \tag{6.55}$$

We see that the differential $d\Phi$ is determined by differentials dP_i and dq_i, i.e. Φ should be treated as a function of P and q: $\Phi \to \Phi(q, P) \equiv \Phi(\{q_i\}, \{P_j\})$. The coefficients in front of dP_i and dq_i are the respective partial derivatives, which results in the following transformation:

$$p_i = \frac{\partial \Phi}{\partial q_i} \quad , \quad Q_i = \frac{\partial \Phi}{\partial P_i} \quad . \tag{6.56}$$

so the Legendre transform reads

$$\Phi(q_i, P_i) \;=\; -F(q, Q) + \sum_i P_i Q_i \;=\; -F\big(q_i, \frac{\partial \Phi}{\partial P_i}\big) + \sum_i P_i \frac{\partial \Phi}{\partial P_i} \tag{6.57}$$

Example: identity transformation $q_i = Q_i$, $p_i = P_i$ The generating function is

$$\Phi_0(q, P) \;=\; \sum_j q_j P_j \quad \Rightarrow \quad p_i = \frac{\partial \Phi_0}{\partial q_i} = P_i, \quad Q_i = \frac{\partial \Phi_0}{\partial P_i} = q_i \tag{6.58}$$

(see Eq. (6.56)).
Example: point transformation.

Take $\Phi = \sum_j f_j(q, t)\, P_j$, then

$$Q_i = \frac{\partial \Phi}{\partial P_i} = f_i(q, t) \;, \tag{6.59}$$

i.e., the new coordinates are functions of only old coordinates (but not momenta): this is a *point* transformation (6.22).

From the first equation in (6.56), we obtain

$$p_i = \frac{\partial \Phi}{\partial q_i} = \sum_j \frac{\partial f_j}{\partial q_i} P_j = \sum_j \frac{\partial Q_j}{\partial q_i} P_j \equiv \sum_j P_j\, a_{ji} \;. \tag{6.60}$$

where $a_{mn} \equiv \frac{\partial Q_m}{\partial q_n}$. Note that since

$$(1)_{jk} \equiv \delta_{jk} = \frac{\partial Q_j}{\partial Q_k} = \sum_i \underbrace{\frac{\partial Q_j}{\partial q_i}}_{a_{ji}} \underbrace{\frac{\partial q_i}{\partial Q_k}}_{(a^{-1})_{ik}} \;, \tag{6.61}$$

the matrix inverse to $a_{ji} \equiv \partial Q_j / \partial q_i$ is given by

$$(a^{-1})_{ik} = \frac{\partial q_i}{\partial Q_k} \;. \tag{6.62}$$

Inverting relation between p_i and P_j, we derive

$$P_j = \sum_i p_i (a^{-1})_{ij} = \sum_i \frac{\partial q_i}{\partial Q_j}\, p_i \;. \tag{6.63}$$

which agrees with

$$P_j = \frac{\partial}{\partial \dot{Q}_j} L\big(q_k(Q,t), \dot{q}_k(Q,t), t\big) = \frac{\partial}{\partial \dot{Q}_j} L\Big(q_k(Q,t), \frac{\partial}{\partial t} q_k(Q,t) + \dot{Q}_i \frac{\partial q_k}{\partial Q_i}, t\Big)$$
$$= \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{\partial q_k}{\partial Q_j} = \sum_k p_k \frac{\partial q_k}{\partial Q_j} \tag{6.64}$$

(cf. Eq. (6.26). Thus, any point transformation (6.59) (plus Eq. (6.64)) is canonical.
Example of a non-canonical transformation:
Take a harmonic oscillator with $H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$, then

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q \ . \tag{6.65}$$

Consider the transformation

$$Q = q_0 \ln \frac{q}{q_0} \quad , \quad P = p_0 \ln \frac{p}{p_0} \ , \tag{6.66}$$

or

$$q = q_0 e^{Q/q_0} \quad , \quad p = p_0 e^{P/p_0} \ . \tag{6.67}$$

The Hamiltonian in the new variables is

$$\tilde{H} = \tilde{H}(P, Q) = \frac{p_0^2}{2m} e^{2P/p_0} + \frac{m\omega^2}{2} q_0^2 e^{2Q/q_0} \ . \tag{6.68}$$

Now, using $Q = q_0 \ln(q/q_0)$, we can find

$$\dot{Q} = q_0 \frac{1}{q/q_0} \cdot \frac{\dot{q}}{q_0} = \frac{q_0}{q} \dot{q} = \frac{q_0}{q} \frac{p}{m} = \frac{q_0}{m} \frac{p}{q} = \frac{q_0}{m} \frac{p_0 e^{P/p_0}}{q_0 e^{Q/q_0}} \ , \tag{6.69}$$

or

$$\dot{Q} = \frac{p_0}{m} e^{P/p_0 - Q/q_0} \ . \tag{6.70}$$

Let us check whether this coincides with $\frac{\partial \tilde{H}}{\partial P}$:

$$\frac{\partial \tilde{H}}{\partial P} = \frac{p_0^2}{2m} \frac{2}{p_0} e^{2P/p_0} = \frac{p_0}{m} e^{2P/p_0} \ . \tag{6.71}$$

Hence, $\dot{Q} \neq \frac{\partial \tilde{H}}{\partial P}$, i.e., the transformation is not canonical.

6.4 Poisson brackets

If $f = f(\{q_i, p_i\}, t)$ and $g = g(\{q_i, p_i\}, t)$ are two functions of dynamical variables p_i and q_i the Poisson bracket $[f, g]_{q,p}$ is defined as

$$[f, g]_{q,p} \stackrel{\text{def}}{=} \sum_k \Big[\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \Big] \tag{6.72}$$

The full time derivative of $f = f(\{q_i, p_i\}, t)$ can be represented as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k} \dot{q}_k + \frac{\partial f}{\partial p_k} \dot{p}_k \right) \tag{6.73}$$

Using Hamilton's equations, we get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q_k} \right) \equiv \frac{\partial f}{\partial t} - \big[H, f\big]_{(q,p)} \tag{6.74}$$

where

$$\big[H, f\big]_{(q,p)} \;=\; \sum_k \left(\frac{\partial H}{\partial q_k}\frac{\partial f}{\partial p_k} - \frac{\partial H}{\partial p_k}\frac{\partial f}{\partial q_k} \right) \tag{6.75}$$

is the Poisson bracket for H and f.

For f to be constant in time, f must satisfy

$$\frac{\partial f}{\partial t} - \big[H, f\big]_{(q,p)} = 0 \;, \tag{6.76}$$

or, if f does not depend on t explicitly, it is constant when $[H, f]_{(q,p)} = 0$.

Examples. Let $g = q_i$. Then

$$\begin{aligned} \big[f, q_i\big]_{(q,p)} &= \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial q_i}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial q_i}{\partial q_k} \right) \\ &= \sum_k \left(\frac{\partial f}{\partial q_k}\cdot 0 - \frac{\partial f}{\partial p_k}\cdot \delta_{ik} \right) = -\frac{\partial f}{\partial p_i} \;. \end{aligned} \tag{6.77}$$

Similarly,

$$\begin{aligned} \big[f, p_i\big]_{(q,p)} &= \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial p_i}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial p_i}{\partial q_k} \right) \\ &= \sum_k \left(\frac{\partial f}{\partial q_k}\cdot \delta_{ik} - \frac{\partial f}{\partial p_k}\cdot 0 \right) = \frac{\partial f}{\partial q_i} \;. \end{aligned} \tag{6.78}$$

and

$$\big[q_k, q_i\big]_{(q,p)} = 0 \;\;,\;\; \big[p_k, p_i\big]_{(q,p)} = 0 \;\;,\;\; \big[q_k, p_i\big]_{(q,p)} = \delta_{ik} \;. \tag{6.79}$$

Using Poisson brackets, we can write Hamilton's equations as

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = -\big[H, q_i\big]_{(q,p)} \;\;\text{or}\;\; \big[H, q_i\big]_{(q,p)} = -\frac{\partial H}{\partial p_i} \;, \tag{6.80}$$

and

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\big[H, p_i\big]_{(q,p)} \;\;\text{or}\;\; \big[H, p_i\big]_{(q,p)} = \frac{\partial H}{\partial q_i} \;, \tag{6.81}$$

in the form explicitly involving only q and p variables.

Properties of Poisson brackets

$$\begin{aligned} [f, c] \;&=\; 0 \qquad \text{if } c \;=\; \text{const} \\ [f_1 + f_2, g] \;&=\; [f_1, g] + [f_2, g] \\ [f_1 f_2, g] \;&=\; [f_1, g] f_2 + f_1 [f_2, g] \\ \big[f, [g, h]\big] \;+\; \big[g, [h, f]\big] \;+\; \big[h, [f, g]\big] \;&=\; 0 \qquad \text{“Jacobi identity”} \end{aligned} \tag{6.82}$$

(First three properties are trivial and the proof of Jacobi identity can be found in Goldstein's textbook).

A consequence of Jacobi identity: if $f(q,p,t)$ and $g(q,p,t)$ are constants of motion, so is $[f,g]$.

Proof:

$$\begin{aligned}\frac{d}{dt}[f,g] &= \frac{\partial}{\partial t}[f,g]-[H,[f,g]] = [\frac{\partial f}{\partial t},g]+[f,\frac{\partial g}{\partial t}]+[f,[g,H]]+[g,[H,f]] = \\ &= [\frac{\partial f}{\partial t}-[H,f],g]+[f,\frac{\partial g}{\partial t}-[H,g]] = 0\end{aligned} \tag{6.83}$$

So, one can construct new constants of motion by taking $[f[f,g]]$, $[g,[g,f]]$, $[f,[f,[f,g]]]$ etc. Since the number of constants of motion is $2n-1$ (with n being a number of generalized coordinates, see Sect. 3.7.4), this process will stop at some point: the new Poisson brackets will be either an old ones or simply constants.

6.4.1 Poisson brackets and canonical transformations

Theorem: the transformation

$$q_i \ \to Q_i(q,p), \qquad p_i \ \to P_i(q,p) \tag{6.84}$$

is a canonical one if and only if

$$[Q_i,Q_j]_{(q,p)} = [P_i,P_j]_{(q,p)} = 0 \qquad \text{and} \qquad [Q_i,P_j]_{(q,p)} = \delta_{ij} \tag{6.85}$$

Proof: see Goldstein's textbook.

Example: harmonic oscillator in terms of P and Q introduced in Eq. (6.51)

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q, \qquad p = \sqrt{2Pm\omega}\cos Q \tag{6.86}$$

The inverse formulas are

$$Q = \arctan m\omega\frac{q}{p}, \qquad P = \frac{m\omega}{2}q^2+\frac{p^2}{2m\omega} \tag{6.87}$$

so ($\arctan' x = \frac{1}{1+x^2}$)

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \frac{1}{1+m^2\omega^2\frac{q^2}{p^2}}+m^2\omega^2\frac{q^2}{p^2}\frac{1}{1+m^2\omega^2\frac{q^2}{p^2}} = 1 \tag{6.88}$$

Theorem: Poisson brackets are canonical invariants , i.e. if

$$q_i \ \to Q_i(q,p), \qquad p_i \ \to P_i(q,p) \tag{6.89}$$

is a canonical transformation, then

$$[f,g]_{(q,p)} = [f,g]_{(Q,P)} \tag{6.90}$$

Proof: if $f(q,p) = f(q(Q,P), p(Q,P))$ and $g(q,p) = g(q(Q,P), p(Q,P))$, by chain rule we get a formula

$$
\begin{aligned}
[f,g]_{q,p} &= \sum_i \Big[\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\Big] \\
&= \sum_i \frac{\partial f}{\partial q_i}\Big[\frac{\partial g}{\partial Q_j}\frac{\partial Q_j}{\partial p_i} + \frac{\partial g}{\partial P_j}\frac{\partial P_j}{\partial p_i}\Big] - \sum_{i,j} \frac{\partial f}{\partial p_i}\Big[\frac{\partial g}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial g}{\partial P_j}\frac{\partial P_j}{\partial q_i}\Big] \\
&= \sum_j \frac{\partial g}{\partial Q_j}[f, Q_j]_{(q,p)} + \sum_j \frac{\partial g}{\partial P_j}[f, P_j]_{(q,p)} \qquad (6.91)
\end{aligned}
$$

Next, take $f(Q,P) = Q_i$ in the above formula:

$$
[Q_i, g]_{(q,p)} = \sum_j \frac{\partial g}{\partial Q_j}[Q_i, Q_j]_{(q,p)} + \sum_j \frac{\partial g}{\partial P_j}[Q_i, P_j]_{(q,p)} = 0 + \frac{\partial g}{\partial P_i} = \frac{\partial g}{\partial P_i} \qquad (6.92)
$$

so we can use $[f, Q_i]_{(q,p)} = -\frac{\partial g}{\partial P_i}$ what follows.

Similarly, taking $f(Q,P) = P_i$ in Eq. (6.91) we get

$$
[P_i, g]_{q,p} = \sum_j \frac{\partial g}{\partial Q_j}[P_i, Q_j]_{(q,p)} + \sum_j \frac{\partial g}{\partial P_j}[P_i, P_j]_{(q,p)} = -\frac{\partial g}{\partial Q_i} \qquad (6.93)
$$

Thus, $[f, Q_i]_{(q,p)} = -\frac{\partial f}{\partial P_i}$ and $[f, P_i]_{q,p} = \frac{\partial f}{\partial Q_i}$ which means that we can rewrite Eq. (6.91) as

$$
[f,g]_{q,p} = \sum_j \frac{\partial g}{\partial Q_j}[f, Q_j]_{(q,p)} + \sum_j \frac{\partial g}{\partial P_j}[f, P_j]_{(q,p)} = \sum_j \Big[\frac{\partial f}{\partial Q_j}\frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_j}\frac{\partial g}{\partial Q_i}\Big] = [f,g]_{(P,Q)} \qquad (6.94)
$$

Thus, Poisson brackets evaluated with one set of canonical variables, have the same value for any other choice of variables related to initial ones by a canonical transformation.

6.5 Canonical transformations and symmetry properties

Consider an infinitesimal canonical transformation

$$
Q_i = q_i + \delta q_i, \qquad P_i = p_i + \delta p_i \qquad (6.95)
$$

In what follows we will keep only linear terms in δq_i and δp_i. Such transformation must have a generating function which differs from identity generating function (6.58) only infinitesimally:

$$
F(q,P) = \sum_j q_j P_j + \epsilon G(q,P) \qquad (6.96)
$$

where ϵ is a small parameter. We get then

$$
\begin{aligned}
p_i &= \frac{\partial F}{\partial q_i} = P_i + \epsilon\frac{\partial G}{\partial q_i} \quad \Leftrightarrow \quad \delta p_i = P_i - p_i = -\epsilon\frac{\partial G}{\partial q_i} \\
Q_i &= \frac{\partial F}{\partial P_i} = q_i + \epsilon\frac{\partial G}{\partial P_i} \quad \Leftrightarrow \quad \delta q_i = Q_i - q_i = \epsilon\frac{\partial G}{\partial p_i} \qquad (6.97)
\end{aligned}
$$

Since $G(q, P) \;=\; G(q, p) + O(\epsilon)$, the above equation can be rewritten as

$$\delta q_i \;=\; \epsilon \frac{\partial G(q,p)}{\partial p_i}, \qquad \delta p_i \;=\; -\,\epsilon \frac{\partial G(q,p)}{\partial q_i} \tag{6.98}$$

so any arbitrary function $G(q, p)$ generates some infinitesimal canonical transformation.

Example: take $G(q, p) = H(q, p)$ and $\epsilon = dt$, then

$$\delta q_i \;=\; dt \frac{\partial H(q,p)}{\partial p_i} \;=\; \dot{q}_i dt, \qquad \delta p_i \;=\; -\,dt \frac{\partial H(q,p)}{\partial q_i} \;=\; \dot{p}_i dt \tag{6.99}$$

We see that $G(q, p) = H(q, p)$ generates the infinitesimal transformation which takes the system at time t and evolves it to time $t + dt$ (because $Q_i = q_i + \delta q_i = q_i + \dot{q}_i dt = q_i(t + dt)$ and similarly for $P_i = p_i(t + dt)$).

Now, the evolution of the system between t_0 and t is generated by a sequence of infinitesimal (canonical) transformations (6.99). The sequence of canonical transformations is also a canonical transformation so one can view a time evolution of the system as being generated by a canonical transformation that takes (q_0, p_0) at time t_0 to (q, p) at time t. This implies the existence of a generating function, and finding of such generating function is equivalent to solving the problem of time evolution of our mechanical system.

Consider a certain function $u(q, p)$.

- Q: what is the change of u under $(q_i, p_i) \;\rightarrow\; (q_i + \delta q_i, p_i + \delta p_i)$?
- A: $\delta u \;=\; u(q_i + \delta q_i, p_i + \delta p_i) - u(q_i, p_i) \;=\; \epsilon[u, G]$

Indeed,

$$\begin{aligned} \delta u \;&=\; u(q_i + \delta q_i, p_i + \delta p_i) - u(q_i, p_i) \;=\; \sum_i \Big[\frac{\partial u}{\partial q_i}\delta q_i + \frac{\partial u}{\partial p_i}\delta p_i\Big] \\ &=\; \epsilon \sum_i \Big[\frac{\partial u}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial G}{\partial q_i}\Big] \;=\; \epsilon[u, G] \end{aligned} \tag{6.100}$$

If we take $u(q, p) = H(q, p)$, then $\delta H = \epsilon[H, G]$ gives the change of H under the infinitesimal transformation generated by $G(q, p)$. Now, if G is a constant of motion $[H, G] = 0$ which means that the canonical transformations generated by G's which are constants of motion leave H invariant. On the other hand, we know that symmetry properties of the system indicate which transformations leave H invariant which means that symmetry defines the set of canonical transformations which leave H invariant.

6.5.1 Total momentum as the generator of spatial translations

Consider an infinitesimal translation of all coordinates of N particles

$$\begin{aligned} \vec{r}_i \;&\rightarrow\; \vec{r}_i' \;=\; \vec{r}_i + \vec{\epsilon} \qquad \text{(same } \vec{\epsilon} \text{ for all } \vec{r}_i), \qquad i \;=\; 1, 2, ...N \\ \vec{p}_i \;&\rightarrow\; \vec{p}_i' \;=\; \vec{p}_i \end{aligned} \tag{6.101}$$

Q: What is the generating function $G(q, p)$ or this transformation?
A: $\vec{G} \;=\; \mathcal{P} \;=\; \sum_i \vec{p}_i$

Indeed, from Eq. (6.98) we see that $\frac{\partial G}{\partial q_i} \sim \delta p_i \;=\; 0$ so $G = G(p_i)$. In addition, $(\delta q_i)_\alpha \;=\; \vec{\epsilon}\cdot \frac{\partial \vec{G}}{\partial (p_i)_\alpha} \;=\; \epsilon_\alpha$ which is the first line in the above equation. Thus, the translations in the α directions are induced by $\mathcal{P}_\alpha$.

Example: two interacting particles

$$H \;=\; \frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} + V(\vec{r}_1 - \vec{r}_2) \tag{6.102}$$

(equal masses for simplicity but no spherical symmetry).

This system is invariant if both particles are translated by $\vec{\epsilon}$ (or equivalently, the frame is translated by $-\vec{\epsilon}$). From Eq. (6.100) we get

$$\begin{aligned}(\delta H)_\alpha \;&=\; [H, \mathcal{P}_\alpha] \;=\; [H, \vec{p}_{1\alpha} + \vec{p}_{2\alpha}] \;=\; \Big[\frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} + V(\vec{r}_1 - \vec{r}_2), \vec{p}_{1\alpha} + \vec{p}_{2\alpha}\Big] \;=\\ &=\; [V(\vec{r}_1 - \vec{r}_2), \vec{p}_{1\alpha} + \vec{p}_{2\alpha}] \;=\; \frac{\partial V(\vec{r}_1 - \vec{r}_2)}{\partial r_{1\alpha}} + \frac{\partial V(\vec{r}_1 - \vec{r}_2)}{\partial r_{2\alpha}} \;=\; 0\end{aligned} \tag{6.103}$$

where we used Eq. (6.78). The meaning of this conservation law becomes evident if one performs the (canonical) transformation to the CM position and relative separation coordinates

$$\begin{aligned}\vec{R} \;&=\; \frac{\vec{r}_1 + \vec{r}_2}{2}, \qquad \vec{r} \;=\; \vec{r}_1 - \vec{r}_2\\ \vec{\mathcal{P}} \;&=\; \vec{p}_1 + \vec{p}_2, , \qquad \vec{p} \;=\; \frac{\vec{p}_1 - \vec{p}_2}{2}\end{aligned} \tag{6.104}$$

In these coordinates

$$H \;=\; \frac{\vec{\mathcal{P}}^2}{4m} + \frac{\vec{p}^{\,2}}{m} + V(\vec{r}) \tag{6.105}$$

which is cyclic in $\vec{R} \quad \Rightarrow \quad \vec{\mathcal{P}} \;= \text{const.}$

6.5.2 Total angular momentum as generator of rotations

Consider a rotation of all coordinates of N particles about the $\hat{n}$ axis by an infinitesimal angle ϵ. From Eq. (2.14)

$$\delta\vec{V}(\vec{r}) \;=\; \epsilon \times \vec{V}(\vec{r}) \tag{6.106}$$

for any vector V so

$$\begin{aligned}\vec{r}_i \;\to\; \vec{r}_i^{\,\prime} \;&=\; \vec{r}_i + \vec{\epsilon} \times \vec{r}_i,\\ \vec{p}_i \;\to\; \vec{p}_i^{\,\prime} \;&=\; \vec{p}_i + \vec{\epsilon} \times \vec{p}_i\end{aligned} \tag{6.107}$$

(again of course same $\vec{\epsilon}$ for all particles). In components Eqs. (6.107) read

$$\begin{aligned}\delta r_{i\alpha} \;&=\; \epsilon\epsilon_{\alpha\beta\gamma}\hat{n}_\beta r_{i\gamma}\\ \delta p_{i\alpha} \;&=\; \epsilon\epsilon_{\alpha\beta\gamma}\hat{n}_\beta p_{i\gamma}\end{aligned} \tag{6.108}$$

The generating function for this transformation is obtained from Eqs. (6.98) so

$$\delta r_{i\alpha} \;=\; \epsilon\epsilon_{\alpha\beta\gamma}\hat{n}_\beta r_{i\gamma} \;=\; \epsilon\frac{\partial G}{\partial p_{i\alpha}} \quad \Rightarrow \quad \frac{\partial G}{\partial p_{i\alpha}} \;=\; \epsilon_{\alpha\beta\gamma}\hat{n}_\beta r_{i\gamma} \tag{6.109}$$

$$\delta p_{i\alpha} \;=\; \epsilon\epsilon_{\alpha\beta\gamma}\hat{n}_\beta p_{i\gamma} \;=\; -\,\epsilon\frac{\partial G}{\partial r_{i\alpha}} \quad \Rightarrow \quad \frac{\partial G}{\partial r_{i\alpha}} \;=\; -\,\epsilon_{\alpha\beta\gamma}\hat{n}_\beta p_{i\gamma} \tag{6.110}$$

which is solved by

$$G(r_i,p_i) \;=\; \sum_j \epsilon_{\mu\nu\lambda}\hat{n}_\mu r_{j\nu}p_{j\lambda} \tag{6.111}$$

Indeed,

$$\begin{aligned}
\frac{\partial G}{\partial r_{i\alpha}} \;&=\; \sum_j \epsilon_{\mu\nu\lambda}\delta_{ij}\delta_{\alpha\nu}\hat{n}_\mu p_{j\lambda} \;=\; \epsilon_{\mu\alpha\lambda}\hat{n}_\mu p_{i\lambda} \;=\; \text{r.h.s. of Eq. (6.110)}\\
\frac{\partial G}{\partial p_{i\alpha}} \;&=\; \sum_j \delta_{ij}\delta_{\alpha\lambda}\epsilon_{\mu\nu\lambda}\hat{n}_\mu r_{j\nu} \;=\; \epsilon_{\mu\nu\lambda}\hat{n}_\mu r_{i\nu} \;=\; \text{r.h.s. of Eq. (6.109)}
\end{aligned} \tag{6.112}$$

Now, note that

$$G(r_i,p_i) \;=\; \sum_j \epsilon_{\mu\nu\lambda}\hat{n}_\mu r_{j\nu}p_{j\lambda} \;=\; \hat{n}_\mu \sum_j \epsilon_{\mu\nu\lambda} r_{j\nu}p_{j\lambda} \;=\; \hat{n}\cdot\vec{L} \tag{6.113}$$

so $\hat{n}\cdot\vec{L}$ is a generator of rotations about $\hat{n}$.

Let $\vec{F}(\vec{r},\vec{p})$ be any vector function of $\vec{r}_i$ and $\vec{p}_i$ (for example $\vec{F} = \sum_j \vec{p}_j = \mathcal{P}$ or $\vec{F} = \vec{r}_i \times \vec{p}_i = \vec{L}_i$). Under rotations on angle ϵ about the $\hat{n}$ axis it changes according to Eq. (6.100)

$$\delta F \;=\; \epsilon[F,\vec{L}\cdot\vec{n}] \tag{6.114}$$

On the other hand, the general formula for rotation of any vector is given by Eq. (6.107): $\delta\vec{F} = \epsilon\hat{n}\times\vec{F}$ which implies that

$$[F,\vec{L}\cdot\vec{n}] \;=\; \hat{n}\times\vec{F} \tag{6.115}$$

If this formula is applied to $\vec{F} = \vec{L}$ and $\hat{n} = \hat{e}_z$ one obtains

$$[\vec{L},L_z] \;=\; \hat{e}_z\times\vec{L} \quad\Leftrightarrow\quad [L_x,L_z] \;=\; -\,L_y, \;\; [L_y,L_z] \;=\; L_x \tag{6.116}$$

Similarly, one can prove that $[L_x,L_y] \;=\; L_z$ so we get

$$[L_\alpha,L_\beta] \;=\; \epsilon_{\alpha\beta\gamma}L_\gamma \tag{6.117}$$

It follows that

- if any two components of $\vec{L}$ are conserved, say L_x and L_y, the remaining component $L_z = [L_x,L_y]$ is also conserved due to Eq. (6.83)
- $[\vec{L}^2,\vec{L}_x] \;=\; [\vec{L}^2,\vec{L}_y] \;=\; [\vec{L}^2,\vec{L}_z] \;=\; 0$

Indeed,

$$\begin{aligned}
[\vec{L}^2,\vec{L}_x] \;&=\; [L_xL_x+L_yL_y+L_zL_z,L_x] \;=\; [L_yL_y,L_x]+[L_zL_z,L_x]\\
&=\; L_y[L_y,L_x]+[L_y,L_x]L_y+L_z[L_z,L_x]+[L_z,L_x]L_z\\
&=\; -\,2L_yL_z+2L_yL_z \;=\; 0
\end{aligned} \tag{6.118}$$

and similarly $[\vec{L}^2,\vec{L}_y] = 0$ and $[\vec{L}^2,\vec{L}_z] = 0$.

Part XIX

6.6 Hamilton-Jacobi theory

Canonical transformations can be used to solve the problem, at least in principle. One way is to find a canonical transformation which makes H cyclic in all coordinates. Another way is to find a canonical transformation that takes $q(t_0), p(t_0)$ to $q(t), p(t)$, then the transformation equations

$$q = q(q_0, p_0, t), \qquad p = p(q_0, p_0, t) \tag{6.119}$$

are the solution of our mechanical problem. Such approach is called Hamilton-Jacobi theory.

Consider a generating function $\Phi(q, P, t)$ yet to be determined. This function generates a canonical Legendre transformation according to Eq. (6.55):

$$p_i = \frac{\partial\Phi(q,P,t)}{\partial q_i}, \qquad Q_i = \frac{\partial\Phi(q,P,t)}{\partial P_i}$$
$$\tilde{H}(Q,P,t) = H(q,p,t) + \frac{\partial\Phi}{\partial t} \tag{6.120}$$

Suppose Φ is such that $\tilde{H} = 0$. If this is the case, the Hamilton equations (6.4) yield

$$\frac{\partial\tilde{H}(Q,P,t)}{\partial P_i} = \dot{Q}_i = 0, \quad \Rightarrow \quad Q_i = \text{const} = Q_{0i} \equiv Q_i(t=0)$$
$$\frac{\partial\tilde{H}(Q,P,t)}{\partial Q_i} = -\dot{P}_i = 0, \quad \Rightarrow \quad P_i = \text{const} = P_{0i} \equiv P_i(t=0) \tag{6.121}$$

This implies that $\Phi(q, P, t)$ is really a function of q_i and t since P_i are constant. Define

$$S(q,t) = \Phi(q, P_0, t)$$

The function S is determined by differential equation (6.120) (with $\tilde{H} = 0$):

$$H\big(q; \frac{\partial S}{\partial q}; t\big) + \frac{\partial S}{\partial t} = 0 \tag{6.122}$$

or, in explicit form,

$$H\big(q_1, ..., q_n; , \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}; t\big) + \frac{\partial S(q_1, ..., q_n; t)}{\partial t} = 0 \tag{6.123}$$

In this way we traded $2n$ first-order coupled differential equations (6.121) for a single partial differential equation with $n+1$ variables (6.121). It is called Hamilton-Jacobi equation. It has $n+1$ constants of integration. This is understood by integrating Eq. (6.123) one variable at a time while keeping the next variable fixed. However, among the $n+1$ integration constants, one is additive

$$S(q_1, ..., q_n; \alpha_1, ..., \alpha_n, \alpha_{n+1}; t) = S(q_1, ..., q_n; \alpha_1, ..., \alpha_n; t) + S_0 \tag{6.124}$$

(where S_0 is the $(n+1)$th constant) since the Eq. (6.123) involves only partial derivatives of S so $S \to S + S_0$ does not affect the equation.

Next, we define *Hamilton's principal function*

$$S(q_1, ..., q_n; P_1, ..., P_n; t) \;=\; S(q_1, ..., q_n; \alpha_1 \to P_1, ..., \alpha_n \to P_n; t) \tag{6.125}$$

and study the canonical transformation generated by $S(q_1, ..., q_n; P_1, ..., P_n; t)$.

From Eq. (6.120) we get

$$\begin{aligned} p_i &= \frac{\partial S(q_j, P_j, t)}{\partial q_i} \\ Q_i &= \frac{\partial S(q_j, P_j, t)}{\partial P_i} \end{aligned} \tag{6.126}$$

Next, from the Hamilton-Jacobi equation (6.123)

$$H\big(q_1, ..., q_n; , \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}; t\big) + \frac{\partial S(q_1, ..., q_n; \alpha_1, ...\alpha_n; t)}{\partial t} \;=\; 0 \tag{6.127}$$

and Eqs. (6.120), (6.126) we see that

$$\begin{aligned} \tilde{H}\big(Q_1, ..., Q_n; P_1, ..., P_n; t\big) \;&=\; H(q_1, ..., q_n; p_1, ..., p_n; t) + \frac{\partial S(q_1, ..., q_n; P_1, ...P_n; t)}{\partial t} \\ =\; H\big(q_1, ..., q_n; , \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}; t\big) &+ \frac{\partial S(q_1, ..., q_n; \alpha_1, ...\alpha_n; t)}{\partial t} \;=\; 0 \end{aligned} \tag{6.128}$$

This, if one solves the Hamilton-Jacobi equation (6.123) one finds the canonical transformation with $\tilde{H} = 0$ leading to conserved P_i and Q_i (see Eq. (6.121)). We already defined $\alpha_i \equiv P_i$ and now we denote

$$Q_i \;=\; \text{const} \;=\; \beta_i$$

The Eq. (6.126) implies

$$\beta_i \;=\; \frac{\partial S(q_1, ..., q_n; \alpha_1, ..., \alpha_n; t)}{\partial \alpha_i} \tag{6.129}$$

which can be inverted to give

$$q_i \;=\; q_i(\alpha_1, ..., \alpha_n; \beta_1, ..., \beta_n; t) \tag{6.130}$$

thus solving our problem.

In practice, to obtain Hamilton-Jacobi equation (6.123) one replaces $p_i \to \frac{\partial S}{\partial q_i}$ and then requires that

$$H\Big(\frac{\partial S}{\partial q_i}, q_i, t\Big) + \frac{\partial S(q_i; t)}{\partial t} \;=\; 0 \tag{6.131}$$

When H does not depend on time explicitly one can use ansatz

$$S(q_1, ...q_n, \alpha_1, ...\alpha_n, t) \;=\; W(q_1, ...q_n, \alpha_1, ...\alpha_n) - \alpha_1 t \tag{6.132}$$

and then the constant α_1 is the energy due to Eq. (6.131)

$$H\Big(\frac{\partial S}{\partial q_i}, q_i\Big) \;=\; -\frac{\partial S(q_i; t)}{\partial t} \;=\; \alpha_1 \tag{6.133}$$

so in terms of W we get

$$H\Big(\frac{\partial W}{\partial q_i}, q_i\Big) \;=\; \alpha_1 \tag{6.134}$$

which is the Hamilton-Jacobi equation for H that does not depend on time explicitly

Let us demonstrate that Hamilton's principal function S can be interpreted as an action along the classical path. Suppose we solved our mechanical problem, namely found q_i in Eq. (6.130) as functions of t and initial conditions. The coordinates q_i define some trajectory in the configuration space of the system. Along this trajectory

$$\frac{dS}{dt} \;=\; \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} \tag{6.135}$$

since $\alpha_1, ..., \alpha_n$ are constants. Moreover, since S satisfies the Hamilton-Jacobi equation (6.123), $\frac{\partial S}{\partial q_i} = p_i$ and $\frac{\partial S}{\partial t} \;=\; -H$, so we get

$$\frac{dS}{dt} \;=\; \sum_i p_i\dot{q}_i - H \;=\; L(q, \dot{q}, t) \tag{6.136}$$

and

$$S(t) \;=\; S[q_1(t), ...q_n(t); \alpha_1, ...\alpha_n; t] \;=\; \int_{t_0}^{t} dt'\; L(t') \;+\; S(t_0) \tag{6.137}$$

which shows that $S(t)$ is the action evaluated along the trajectory (6.130). Unfortunately, the equation (6.137) is useless in determining S since it implies *a priori* knowledge of the trajectory.

6.6.1 Example 1: harmonic oscillator

Let us take

$$H \;=\; \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \tag{6.138}$$

The Eq. (6.131) takes the form

$$\frac{1}{2m}\Big(\frac{\partial S(q,\alpha,t)}{\partial q}\Big)^2 + \frac{m\omega^2 q^2}{2} + \frac{\partial S(q,\alpha,t)}{\partial t} \;=\; 0 \tag{6.139}$$

for the harmonic oscillator. (The additive constant S_0 is ignored here since it is irrelevant for the solution). Since t does not appear in H explicitly, we can try the solution in the form

$$S(q,\alpha,t) \;=\; W(q,\alpha) - \alpha t \tag{6.140}$$

We get

$$\frac{1}{2m}\Big(\frac{\partial W(q,\alpha)}{\partial q}\Big)^2 + \frac{m\omega^2 q^2}{2} \;=\; \alpha \tag{6.141}$$

which has a solution

$$W(q,\alpha) \;=\; \pm\sqrt{2m}\int^{q} dq'\;\sqrt{\alpha - \frac{1}{2}m\omega^2 q'^2} \tag{6.142}$$

This integral can be easily calculated, but it is of no interest, since the trajectory $q(\alpha, \beta, t)$ is determined by Eq. (6.129)

$$\begin{aligned}
\beta \;&=\; \frac{\partial S}{\partial \alpha} \;=\; \pm\sqrt{\frac{m}{2}}\int^q \frac{dq'}{\sqrt{\alpha - \frac{1}{2}m\omega^2 q'^2}} \;-\; t \\
\Leftrightarrow\; \beta + t \;&=\; \pm\sqrt{\frac{m}{2}}\int^q \frac{dq'}{\sqrt{\alpha - \frac{1}{2}m\omega^2 q'^2}}
\end{aligned} \tag{6.143}$$

Let us take $(-)$ sign, then

$$\beta + t \;=\; \frac{1}{\omega}\arccos\left(q\omega\sqrt{\frac{m}{2\alpha}}\right) \quad\Rightarrow\quad q \;=\; \sqrt{\frac{2\alpha}{m\omega}}\cos\omega(t+\beta) \tag{6.144}$$

where α and β are determined now by the initial conditions. Note that the constant α is actually the energy since from Eq. (6.131) and Eq. (6.140) we get

$$H \;=\; \alpha \tag{6.145}$$

As an example, let us take initial conditions $q(0) = q_0$ and $\dot{q}(0) = 0$, then $\alpha = \frac{m\omega^2}{2}q_0^2$ and $\beta = 0$:

$$\begin{aligned}
&q(t) \;=\; q_0\cos\omega(t+\beta) \;\Rightarrow\; \dot{q}(t) \;=\; -\,\omega q_0 \sin\omega(t+\beta) \;\Rightarrow\; \sin\omega\beta = 0 \;\Rightarrow\; \beta = 0 \\
&\Rightarrow\; q(t) \;=\; q_0\cos\omega t
\end{aligned} \tag{6.146}$$

The function S can be found from Eq. (6.140) and (6.142)

$$S \;=\; -\,m\omega\int^q dq'\sqrt{q_0^2 - q'^2} - \frac{m\omega^2 q_0^2}{2}t \;=\; m\omega^2 q_0^2\int^t dt'\sin^2\omega t' - \frac{m\omega^2 q_0^2}{2}t \tag{6.147}$$

On the other hand, the action is given by

$$\begin{aligned}
S \;&=\; \int^t dt'\left(\frac{m}{2}\dot{q}^2(t') - \frac{m\omega^2}{2}q^2(t')\right) \;=\; \frac{m\omega^2 q_0^2}{2}\int^t dt'(\sin^2\omega t' - \cos^2\omega t') \\
&=\; \frac{m\omega^2 q_0^2}{2}\int^t dt'(2\sin^2\omega t' - 1) \;=\; -\,\frac{m\omega^2 q_0^2}{2}t + m\omega^2 q_0^2\int^t dt'\sin^2\omega t'
\end{aligned} \tag{6.148}$$

6.6.2 Example 2: particle in a central potential

The Hamilton-Jacobi equations are useful for the class of problems which admit separation of variables. As an example, we will consider the motion of a particle in a plane under the influence of a central force. The Lagrangian is

$$L \;=\; \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \tag{6.149}$$

The canonical momenta are

$$p_r \;=\; \frac{\partial L}{\partial \dot{r}} \;=\; m\dot{r}, \qquad p_\phi \;=\; \frac{\partial L}{\partial \dot{\phi}} \;=\; mr^2\dot{\phi} \tag{6.150}$$

so the Hamiltonian has the form

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r) \tag{6.151}$$

As we discussed above, since this Hamiltonian does not depend on time explicitly we can try the ansatz

$$S(q_1, q_2; \alpha_1, \alpha_2; t) = W(q_1, q_2; \alpha_1, \alpha_2; t) - \alpha_1 t \tag{6.152}$$

and the Hamilton-Jacobi equation (6.123) turns to

$$\alpha_1 = H(r, \phi, \frac{\partial W}{\partial r}, \frac{\partial W}{\partial \phi}, \alpha_1, \alpha_2) \quad \Rightarrow \quad \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \phi}\right)^2 + V(r) = \alpha_1 \tag{6.153}$$

Let us try now ansatz with separation of variables

$$W(r, \phi, \alpha_1, \alpha_2) = W_1(r, \alpha_1, \alpha_2) + W_2(\phi, \alpha_1, \alpha_2) \tag{6.154}$$

We get

$$\begin{aligned} \frac{1}{2m}\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W_2}{\partial \phi}\right)^2 + V(r) &= \alpha_1 \\ \Rightarrow \left(\frac{\partial W_2}{\partial \phi}\right)^2 &= r^2\left[2m[\alpha_1 - V(r)] - \left(\frac{\partial W_1}{\partial r}\right)^2\right] \end{aligned} \tag{6.155}$$

The l.h.s. depends only on ϕ while the r.h.s. only on r so both of them must be constant. Let us choose this constant as α_2^2, then $\frac{\partial W_2}{\partial \phi} = \alpha_2$ and hence the Eq. (6.154) reads

$$W(r, \phi, \alpha_1, \alpha_2) = W_1(r, \alpha_1, \alpha_2) + \alpha_2 \phi \tag{6.156}$$

One could have guessed this form for W by observing that H is cyclic in ϕ so

$$p_\phi = \frac{\partial S}{\partial \phi} = \frac{\partial W}{\partial \phi} = \text{const} \equiv \alpha_2 \tag{6.157}$$

leading to Eq. (6.156).

Next, using Eq. (6.156) we can rewrite the Hamilton-Jacobi equation (6.155) as

$$\begin{aligned} \frac{1}{2m}\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{\alpha_2^2}{2mr^2} + V(r) &= \alpha_1 \\ \Rightarrow \frac{\partial W_1}{\partial r} &= \sqrt{2m[\alpha_1 - V(r)] - \frac{\alpha_2^2}{r^2}} \end{aligned} \tag{6.158}$$

and we get

$$W(r, \phi, \alpha_1, \alpha_2) = \int^r dr' \sqrt{2m[\alpha_1 - V(r')] - \frac{\alpha_2^2}{r'^2}} + \alpha_2 \phi \tag{6.159}$$

Now, from Eq. (6.129) we can find β_1 and β_2

$$\begin{aligned} \beta_1 &= \frac{\partial S}{\partial \alpha_1} = \frac{\partial W}{\partial \alpha_1} - t \\ \beta_2 &= \frac{\partial S}{\partial \alpha_2} = \frac{\partial W}{\partial \alpha_2} \end{aligned} \tag{6.160}$$

so we get

$$t + \beta_1 = m\int^r dr' \frac{1}{\sqrt{2m[\alpha_1 - V(r') - \alpha_2^2 r'^{-2}}} \tag{6.161}$$

and

$$\phi = \beta_2 + \alpha_2 \int^r \frac{dr'}{r'^2} \frac{1}{\sqrt{2m[\alpha_1 - V(r')] - \alpha_2^2 r'^{-2}}} \tag{6.162}$$

which are Eqs. (1.68) and (1.70) for the trajectory in a central potential. We see now that the constant α_2 is the conserved angular momentum (and α_1 is energy as we saw above).

6.7 Action-angle variables

Hamilton-Jacobi theory is very convenient for description of conservative systems whose motion is both separable and periodic. Separability means that

$$W(q_1, ...q_n; \alpha_1, ...\alpha_n) = W_1(q_1; \alpha_1, ...\alpha_n) + W_2(q_2; \alpha_1, ...\alpha_n) + ... + W_n(q_n; \alpha_1, ...\alpha_n) \tag{6.163}$$

and two types of periodic motion are *libration* (Fig. a) and *rotation* (Fig. b).

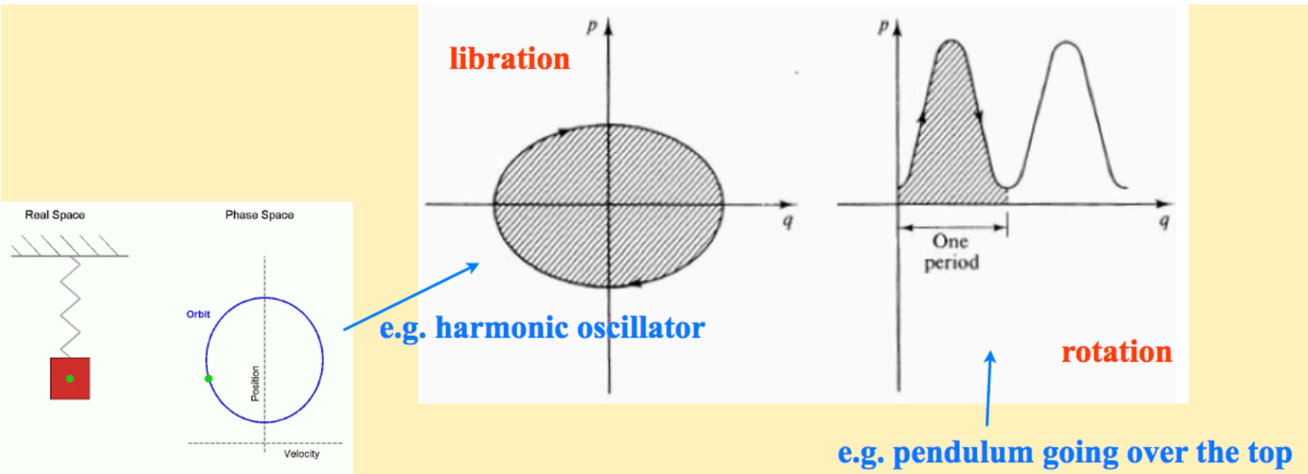

Figure 62. Libration and rotation

The example of libration is a simple pendulum with small oscillations; the example of rotation is the pendulum with enough energy to go over the top.

The action variables are defined as

$$J_i = \oint p_i dq_i \tag{6.164}$$

The integral is the area in phase space taken over one period of the motion.

From Eqs. (6.126), (6.132) and (6.163) we get

$$p_i = \frac{\partial W}{\partial q_i} = \frac{\partial W_i(q_i; \alpha_1, ..\alpha_n)}{\partial q_i} \tag{6.165}$$

so the Eq. (6.164) gives J_i as functions of α's

$$J_i \;=\; J_i(\alpha_1, ..\alpha_n) \tag{6.166}$$

Thus, J_i are constants of motion (since α_i are). We assume that Eqs. (6.166) can be inverted

$$\alpha_i \;=\; \alpha_i(J_1, ..J_n) \tag{6.167}$$

As we saw in Eq. (6.133), the constant α_1 is the energy

$$\alpha_1 \;=\; \alpha_1(J_1, ..J_n) \;=\; H(J_1, ..J_n) \;=\; E \tag{6.168}$$

Now we can take J_i as new integration constants in place of α_i:

$$\begin{aligned} W[q_1, ...q_n; \alpha_1(J), ...\alpha_n(J)] \;&=\; \bar{W}[q_1, ...q_n; J_1, ...J_n) \\ S[q_1, ...q_n; \alpha_1(J), ...\alpha_n(J); t] \;&=\; \bar{S}[q_1, ...q_n; J_1, ...J_n; t) \;=\; \bar{W} - t\alpha_1(J) \end{aligned} \tag{6.169}$$

where bar is added for convenience.

Now let us study canonical transformation (6.126) generated by Hamilton's principal function (6.169) with the canonical momenta being J_i

$$\bar{S}[q_1, ...q_n; P_1, ...P_n; t) \;=\; \bar{S}[q_1, ...q_n; J_1, ...J_n; t), \qquad P_i \equiv J_i \tag{6.170}$$

Rewriting Eq. (6.126) we get

$$\begin{aligned} p_i \;&=\; \frac{\partial \bar{S}(q_j, P_j, t)}{\partial q_i}\Big|_{\alpha_k=\text{const}} \;=\; \frac{\partial \bar{S}(q_j, J_j, t)}{\partial q_i}\Big|_{J_k=\text{const}} \\ Q_i \;&=\; \frac{\partial \bar{S}(q_j, P_j, t)}{\partial P_i} \;=\; \frac{\partial \bar{S}(q_j, J_j, t)}{\partial J_i} \end{aligned} \tag{6.171}$$

In addition, it is easy to check that $\bar{S}[q_1, ...q_n; J_1, ...J_n; t)$ satisfies the Hamilton-Jacobi equation (6.123)

$$\begin{aligned} -\frac{\partial \bar{S}(q_1, ..., q_n; t)}{\partial t} \;&=\; \alpha_1 \;=\; \bar{H}(J_1, ...J_n) \\ &=\; H\big(q_1, ..., q_n; p_1, ...p_n\big) \;=\; H\big(q_1, ..., q_n; , \frac{\partial \bar{S}}{\partial q_1}, ..., \frac{\partial \bar{S}}{\partial q_n}\big) \end{aligned} \tag{6.172}$$

Since $\bar{S}$ satisfies the Hamilton-Jacobi equation the canonical transformation (6.126) generated by $\bar{S}$ leads to $\tilde{H} = 0$ and conserved $P_i = J_i$ (we already saw that) and conserved Q_i:

$$\begin{aligned} \bar{P}_i \;\equiv\; J_i \;&=\; \text{const} \\ \bar{Q}_i \;&\equiv\; \bar{\beta}_i \end{aligned} \tag{6.173}$$

Next, we define *angle* variables

$$w_i(q_1, ...q_n; J_1, ...J_n) \;\equiv\; \frac{\partial}{\partial J_i}\bar{W}(q_1, ...q_n; J_1, ...J_n) \tag{6.174}$$

Since $\bar{\beta}_i = Q_i = \frac{\partial \bar{S}(q_j, J_j, t)}{\partial J_i}$ are constants of motion, from Eq. (6.169) ($\bar{S} = \bar{W} - t\alpha_1(J)$) we get

$$\begin{aligned}
\bar{\beta}_i(q_1, ...q_n; J_1, ...J_n) &= \frac{\partial \bar{S}(q_1, ...q_n; J_1, ...J_n; t)}{\partial J_i} \\
&= \frac{\partial \bar{W}(q_1, ...q_n; J_1, ...J_n)}{\partial J_i} - t\frac{\partial \alpha_1(q_1, ...q_n; J_1, ...J_n)}{\partial J_i} \\
&= w_i(q_1, ...q_n; J_1, ...J_n) - t\frac{\partial \alpha_1(q_1, ...q_n; J_1, ...J_n)}{\partial J_i} = \text{const}
\end{aligned} \tag{6.175}$$

and therefore

$$w_i = \nu_i t + \bar{\beta}_i \tag{6.176}$$

where the "frequency" ν_i is defined as

$$\nu_i \equiv \frac{\partial}{\partial J_i}\alpha_1(J_1, ...J_n) = \frac{\partial}{\partial J_i}H(J_1, ...J_n) \tag{6.177}$$

Note that each w_i increase linearly since $\bar{\beta}_i$ and ν_i are constants of motion.

Now consider periodic motion so the system returns to initial configuration after an integer number of periods. Let us find change in angle variables after the system goes over large integer number of periods. The infinitesimal change of w_i is

$$dw_i = \sum_j \frac{\partial w_i}{\partial q_j} dq_j = \frac{\partial}{\partial J_i}\sum_j \frac{\partial W}{\partial q_j} dq_j \tag{6.178}$$

where we used Eq. (6.174) and the fact that J's are constants of motion. Next, we recall that we assumed separability (6.163) and get

$$dw_i = \frac{\partial}{\partial J_i}\sum_j \frac{\partial W_j}{\partial q_j} dq_j = \frac{\partial}{\partial J_i}\sum_j p_j dq_j \tag{6.179}$$

where we used Eq. (6.165).

Now let us integrate Eq. (6.179) over one period Δt of the system during which each degree of freedom q_k undergoes integer number n_k of periods τ_k.

$$\Delta t = n_k \tau_k, \qquad k = 1, 2, ...n \tag{6.180}$$

From Eq. (6.176) we get

$$\Delta w_k = \nu_k \Delta t = \nu_k n_k \tau_k \tag{6.181}$$

On the other hand, the total change in w_k variable can be obtained by integrating Eq. (6.179)

$$\Delta w_i = \frac{\partial}{\partial J_i}\sum_j \int p_j dq_j \tag{6.182}$$

Each degree of freedom w_j has executed an integer number of periods n_j so

$$\int p_j dq_j = n_j \oint p_j dq_j = n_j J_j \tag{6.183}$$

(recall the definition of action variables (6.164)) and we get

$$\Delta w_i \;=\; \frac{\partial}{\partial J_i} \sum_j n_j J_j \;=\; n_i \tag{6.184}$$

Comparing this equation to Eq. (6.181), we see that

$$\nu_i \;=\; \frac{1}{\tau_i} \tag{6.185}$$

which justifies the name "frequency" for ν_i. They are called *fundamental frequencies* of the system. From Eq. (6.177) we see that they are partial derivatives of the energy with respect to action variables J_i.

6.7.1 Example: harmonic oscillator in two dimensions

Consider 2-dim harmonic oscillator with two different spring constants. The Hamiltonian is

$$H \;=\; \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k_1}{2} q_1^2 + \frac{k_2}{2} q_2^2 \;=\; \alpha \tag{6.186}$$

(change of name $\alpha_1 \to \alpha$) and the Hamilton-Jacobi equation (6.134) becomes

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2\right] + \frac{k_1}{2} q_1^2 + \frac{k_2}{2} q_2^2 \;=\; \alpha \tag{6.187}$$

Separation of variables

$$W(q_1, q_2) \;=\; W_1(q_1) + W_2(q_2) \tag{6.188}$$

leads to

$$\left[\frac{1}{2m}\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \frac{k_1}{2} q_1^2\right] + \left[\left(\frac{\partial W_2}{\partial q_2}\right)^2 + \frac{k_2}{2} q_2^2\right] \;=\; \alpha \tag{6.189}$$

which means that two expressions in square brackets are constants:

$$\begin{aligned} \frac{1}{2m}\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \frac{k_1}{2} q_1^2 \;&=\; \alpha_1 \\ \frac{1}{2m}\left(\frac{\partial W_2}{\partial q_2}\right)^2 + \frac{k_2}{2} q_2^2 \;&=\; \alpha_2 \\ \alpha_1 + \alpha_2 \;=\; \alpha \;&=\; E \end{aligned} \tag{6.190}$$

These equatioin can be easily solved:

$$\begin{aligned} \frac{dW_1(q_1)}{dq_1} \;&=\; p_1 \;=\; \pm\sqrt{m(2\alpha_1 - k_1 q_1^2)} \\ \frac{dW_2(q_2)}{dq_2} \;&=\; p_2 \;=\; \pm\sqrt{m(2\alpha_2 - k_2 q_2^2)} \end{aligned} \tag{6.191}$$

where we used Eq. (6.171)

$$p_i \;=\; \frac{\partial \bar{S}(q_j, J_j, t)}{\partial q_i}\bigg|_{J_k = \text{const}} \;=\; \frac{dW_i(q_1, q_2)}{dq_i} \tag{6.192}$$

Now introduce new variables θ

$$
\begin{aligned}
q_1 &= \sqrt{\frac{2\alpha_1}{k_1}}\sin\theta_1 \qquad \Rightarrow \qquad dq_1 = \sqrt{\frac{2\alpha_1}{k_1}}\cos\theta_1 d\theta_1 \\
q_2 &= \sqrt{\frac{2\alpha_2}{k_2}}\sin\theta_2 \qquad \Rightarrow \qquad dq_2 = \sqrt{\frac{2\alpha_2}{k_2}}\cos\theta_2 d\theta_2
\end{aligned} \tag{6.193}
$$

The action variables (6.164) take the form

$$
\begin{aligned}
J_1 &\equiv \oint p_1 dq_1 = 2\alpha_1\sqrt{\frac{m}{k_1}}\int_0^{2\pi} d\theta_1 \ \cos^2\theta_1 = 2\pi\alpha_1\sqrt{\frac{m}{k_1}} \\
J_2 &\equiv \oint p_2 dq_2 = 2\alpha_2\sqrt{\frac{m}{k_2}}\int_0^{2\pi} d\theta_2 \ \cos^2\theta_2 = 2\pi\alpha_2\sqrt{\frac{m}{k_2}}
\end{aligned} \tag{6.194}
$$

so

$$
\alpha_1 = \frac{J_1}{2\pi}\sqrt{\frac{k_1}{m}}, \qquad \alpha_2 = \frac{J_2}{2\pi}\sqrt{\frac{k_2}{m}} \tag{6.195}
$$

and the Hamiltonian defined by Eqs. (6.186) and (**??**) takes the form

$$
H(J_1, J_2) = \frac{J_1}{2\pi}\sqrt{\frac{k_1}{m}} + \frac{J_2}{2\pi}\sqrt{\frac{k_2}{m}} \tag{6.196}
$$

From this Hamiltonian we can obtain fundamental frequencies using Eq. (6.177)

$$
\begin{aligned}
\nu_1 &= \frac{\partial H}{\partial J_1} = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}} \\
\nu_2 &= \frac{\partial H}{\partial J_2} = \frac{1}{2\pi}\sqrt{\frac{k_2}{m}}
\end{aligned} \tag{6.197}
$$

which are frequencies of independent oscillations in the first and second coordinates.