Physics 603 Classical Mechanics - Spring 2020
Minkowski space and relativistic dynamics I

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1 3+1 Minkowski space

Vectors in Minkowski space have four components, which are indicated by a greek superscript v^μ with $\mu = 0, 1, 2, 3$. A latin index is often used to distinguish the spatial part of the vector v^k with $k = 1, 2, 3$. The four-vector

$$x^\mu = (ct, \mathbf{x}), \tag{1.1}$$

represents an **event** which specifies the **position** where something happens and the **time** when it happens. Another example is the energy-momentum four-vector

$$p^\mu = (E/c, \mathbf{p}) = m_0 c(\gamma, \gamma\beta). \tag{1.2}$$

In general, four-vectors in Minkowski space must transform properly under Lorentz Transformations (boosts and rotations).

1.1 Scalar product

The inner product g of any four-vectors is an invariant

$$g(v^\mu, u^\nu) = g(v'^\mu, u'^\nu) = s. \tag{1.3}$$

For two events x^μ, y^μ the scalar product is given by

$$g(x^\mu, y^\mu) = (ct_x)(ct_y) - \mathbf{x} \cdot \mathbf{y} = x^0 y^0 - x^k y^k, \tag{1.4}$$

where the sum over $k = 1, 2, 3$ is implied. For the four-momentum we notice that the inner product with itself is proportional to the squared rest mass

$$g(p^\mu, p^\mu) = \frac{E^2}{c^2} - \mathbf{p}^2 = m_0^2 c^2. \tag{1.5}$$

The scalar product of the difference between two events dx with itself returns the **invariant distance**

$$g(dx^\mu, dx^\mu) = ds^2 = d(ct)^2 - d\mathbf{r}^2. \tag{1.6}$$

Two events can be classified according to their invariant distance as

1. **Light-like:** $ds^2 = 0$ 2. **Space-like:** $ds^2 < 0$ 3. **Time-like:** $ds^2 > 0$

The **eigentime**, or **proper time**, for a time-like separation (distance) is defined as

$$d\tau = \sqrt{ds^2}/c = \gamma^{-1} dt. \tag{1.7}$$

1.2 Four-forms

A four-form (or covariant vector) is a functional U that maps a Minkowski four-vector (or contravariant vector) into scalars

$$U(x^\mu) = s. \tag{1.8}$$

Like vectors, forms have four components indicated by a lower index u_μ such that

$$U(v^\mu) = u_\mu v^\mu. \tag{1.9}$$

1.3 Tensors

Tensors are objects with more than one upper or lower index

$$T^{\mu\nu\ldots}_{\alpha\beta\ldots}, \tag{1.10}$$

these can be constructed out of vectors and forms. The Lorentz matrix itself represents a tensor which takes a vector in a frame S and returns a vector in a different frame S', thus

$$S': \qquad x'^\mu = \Lambda^\mu_\nu x^\nu. \tag{1.11}$$

For example, a boost in the z direction is given by

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}. \tag{1.12}$$

The **metric tensor** g

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.13}$$

allows us to write the scalar product of two vectors as

$$g(x^\mu, y^\nu) = g_{\mu\nu} x^\mu y^\nu. \tag{1.14}$$

On the other hand, the action of the metric tensor on a vector returns a form, thus lowers its index

$$g_{\mu\nu} x^\nu = x_\mu = (ct, -\mathbf{x}). \tag{1.15}$$

2 Relativistic formulation

Upon having defined the proper objects of the Minkowski (scalars, vectors, forms, and tensors) now it is possible to rewrite the laws of physics in a relativistic formulation. The non-relativistic coordinates are replaced by events x^μ, with the **four-velocity** given by the derivative with respect to the eigentime

$$\frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (\gamma c, \mathbf{u}), \tag{2.1}$$

such that

$$m_0 u^\mu = (\gamma m_0 c, \gamma m_0 \mathbf{u}) = (E/c, \mathbf{p}) = p^\mu. \tag{2.2}$$

Second Newton's law can then be written in the relativistic formulation as

$$K^\mu = \frac{dp^\mu}{d\tau} = \gamma \frac{dp^\mu}{dt}, \tag{2.3}$$

where K^μ represents the **four-force**. This becomes even more evident by defining the **pseudo-force** $F^\mu = K^\mu/\gamma$

$$F^\mu = \frac{dp^\mu}{dt} = \left(\frac{dp^0}{dt}, \mathbf{F} \right) = \left(\frac{1}{c} \frac{dE}{dt}, \mathbf{F} \right), \tag{2.4}$$

where we can identify the power of the system in the zero component of the pseudo-force. In fact, the scalar product between the pseudo-force and the four-velocity returns the expression for the power in terms of the spatial velocity

$$\frac{dp^\mu}{dt} u_\mu = \gamma \left(\frac{dE}{dt} - \mathbf{F} \cdot \mathbf{u} \right) = \frac{m_0}{2} \frac{d}{dt} u^\mu u_\mu = \frac{m_0}{2} \frac{d}{dt} c^2 = 0 \quad \Rightarrow \quad \frac{dE}{dt} = \mathbf{F} \cdot \mathbf{u}. \tag{2.5}$$