Graduate Classical Mechanics
PHYS 603, Spring 2019 Dr. Sebastian E. Kuhn

Lecture Participation

Submitted by

Sunil Pokharel
Department of Physics
Old Dominion University

February 8, 2019

Conserved Quantities and Symmetry

The *Euler- Langrange* eqution of motion is :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{1}$$

where q_i's are generalized co-ordinates, $\dot{q}_i$'s are generalized velocity. The *Langrangian* $\mathcal{L}$ is defined as the kinetic energy (T) minus potential energy (V)

$$\mathcal{L} = T - V \tag{2}$$

The sufficient information about the motion of a system can be gathered even without the complete solution of these equations by knowing the physical nature of the system motion. Conservation theorems for a system provide the constants of motion which help in describing the motion of the system.
The *generalized momentum* or *cannonical momentum* can defined as:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{3}$$

Then, the Eq.(1) can be written as:

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} \tag{4}$$

If Langrangian $\mathcal{L}$ does not depend explicitly on some particular generalized coordinate q_i, then $\frac{\partial \mathcal{L}}{\partial q_i} = 0$, and the corresponding generalized momentum p_i is constant. Such coordinates are said to be *cyclic* or *ignorable*. That means generalized momenta corresponding to cyclic coordinates are constants of motion. Note that if q_i is Cartesian coordinate, p_i is the linear momentum but in general, p_i does not necessarily have the dimensions of a linear momentum.
The existence of conserved quantities has an important relationship to the symmetry of the problem. If the system is invariant under some continuous transformation, then langrangian $\mathcal{L}$ (or T and V) are unchanged by alterations in the corresponding generalized coordinate q_j. That means, $\frac{\partial \mathcal{L}}{\partial q_i} = 0$ for this particular q_i and the Eq. (4) shows that the momemtum p_i is constant of motion.

Again let us define an energy function h

$$h(q_i, \dot{q}_i, t) = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t) = \sum_i p_i \dot{q}_i - \mathcal{L} \quad (5)$$

where, $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is the canonical momentum of the system. Now, differentiating with respect to time, we obtain

$$\frac{dh}{dt} = \sum_i \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} - \frac{\partial \mathcal{L}}{\partial q_i}\frac{q_i}{dt} - \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right] - \frac{\partial \mathcal{L}}{\partial t}$$

$$\therefore \quad \frac{dh}{dt} = -\frac{\partial \mathcal{L}}{\partial t} \quad (6)$$

If the langrangian is not an explicit function of time, the Eq.(6) says that the value of the function h is conserved along the trajectory of the system. The function h may or may not be equal to the total energy of the system (it is usually equal to the kinetic plus potential energy of the system IF the Lagrangian contains only terms quadratic in the velocities).

Noether's Theorem: "For each symmetry of the Lagrangian, there is a conserved quantity".

We shall discuss these ideas with the following examples:

a. Motion of two body system

Two masses m_1 and m_2 moving under their mutual gravitational attraction in a uniform external gravitational field whose acceleration is g. Choosing coordinates the Cartesian coordinates X, Y, Z of the center of mass vector $\mathbf{R}$ (taking Z in the direction of g) and the spherical coordinates r, θ and ϕ that define the relative vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ from m_1 and m_2.

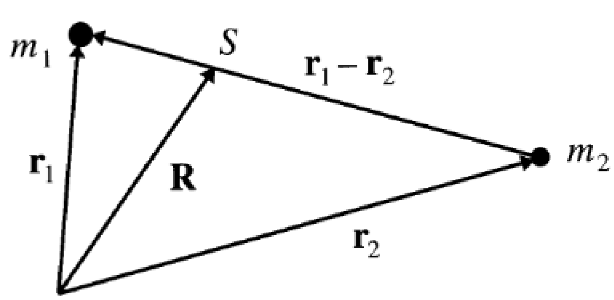

Figure 1: Center of gravity (S) and relative coordinates of two masses m_1 and m_2

The Kinetic energy K, and Potential energy V of the system are:

$$K = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\ \dot{\phi}^2\right)$$
$$V = m_1 g z_1 + m_2 g z_2 - G\frac{m_1 m_2}{r} = MgZ - \frac{G\mu M}{r} \tag{7}$$

where $M = m_1 + m_2$, and $\mu = \dfrac{m_1\ m_2}{m_1 + m_2}$ is the reduced mass of the system

Now, the Langrangian of the system is:

$$\mathcal{L} = T - V$$
$$\mathcal{L} = \frac{M}{2}\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - MgZ + \frac{GM\mu}{r} \tag{8}$$

Since, the coordinates X, Y and ϕ are cyclic, the the system is invariant under a shift of the coordinate axes in the XY-plane and rotation around Z-axis; equivalently

$$\frac{\partial\mathcal{L}}{\partial X} = \frac{\partial\mathcal{L}}{\partial Y} = 0; \quad \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{9}$$

so that the corresponding momenta p_X, p_Y, and p_ϕ are constants. Using Eq.(4), we obtain ,

$$p_X = \frac{\partial\mathcal{L}}{\partial\dot{X}} = M\dot{X}, \quad p_Y = \frac{\partial\mathcal{L}}{\partial\dot{Y}} = M\dot{Y} \ \ and \ \ p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \mu r^2 sin^2\theta\ \dot{\phi} \tag{10}$$

And the relations

$$\dot{p}_X = \dot{p}_Y = 0 \ \text{ and } \ \dot{p}_\phi = 0 \tag{11}$$

merely express the conservation of the linear momenta p_X and p_Y, and angular momentum p_ϕ.

b. Mass spring system in polar coordinates

A massless spring of force constant k and natural length l lies on a horizontal frictionless table. The spring is attached to the table at one end (the origin O), and can rotate freely around it. An object of mass m is attached to the other end of the spring as shown in Fig.2.

Using the polar coordinate system ($\mathbf{r}$, ϕ), the Lagrangian of the system is:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\ \dot{\phi}^2) - \frac{1}{2}\ k(r - l)^2 \tag{12}$$

which is obviously invariant under a rotation of the coordinate axes about the z-axis. Correspondingly, the variable ϕ is cyclic, so that p_ϕ is a constant of the motion.

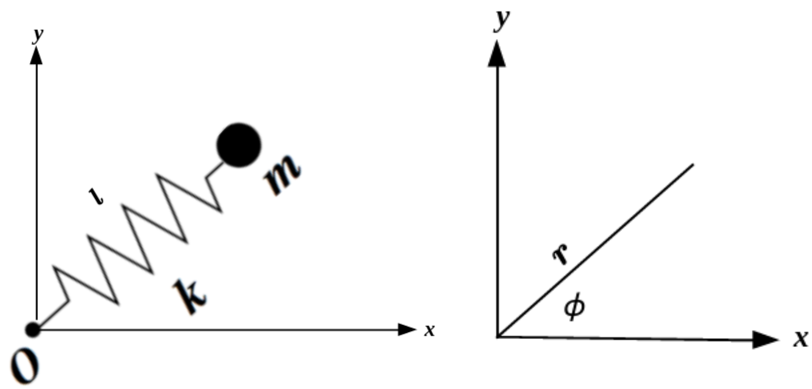

Figure 2: A mass-spring system in xy-plane

Then, from Euler- Langrange equation for the variable ϕ yields:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = \frac{d}{dt}(mr^2\dot{\phi}) = 0$$

$$\therefore p_\phi = mr^2\dot{\phi} = L_z = constant \tag{13}$$

is just the magnitude of the angular momentum L_z, which is conserved. In this case

$$\dot{\phi} = \frac{p_\phi}{mr^2}$$

$$\phi = \phi_0 + \int \frac{p_\phi}{mr^2}dt \tag{14}$$

is the solution for angualr variable ϕ. Again, the EL equation for the radial coordinate r yields:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\phi}^2 + k(r-l) = 0$$

$$m\ddot{r} - \frac{p_\phi^2}{mr^3} + k(r-l) = 0 \tag{15}$$

In this particular example, the langrangian does not depend explicitly on time, and the energy function h is equal to the total energy E of the system:

$$\begin{aligned} E = T + V &= \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\phi}^2 + \frac{k}{2}(r-l)^2 \\ &= \frac{m}{2}\dot{r}^2 + \frac{p_\phi^2}{2mr^2} + \frac{k}{2}(r-l)^2 = T + \text{“}V\text{”} \\ \therefore\ E &= T + \text{“}V\text{”} = \text{constant}, \end{aligned} \tag{16}$$

$$\text{where ‘}V\text{”} = \frac{p_\phi^2}{2mr^2} + \frac{k}{2}(r-l)^2 \ \text{ is pseudo -potential}$$

Now, Newton's law becomes:

$$m\ddot{r} = -\frac{\partial\text{“}V\text{”}}{\partial r} = F \tag{17}$$

At equilibrium: $r = r_0$, and

$$\begin{aligned} \frac{\partial\text{“}V\text{”}}{\partial r} &= F = 0 \\ -\frac{p_\phi^2}{mr^3} + k(r-l) &= 0 \qquad \text{and} \quad p_\phi^2 = mk(r^4 - lr^3) \\ -\frac{m^2r^4\dot{\phi}^2}{mr^3} + k(r-l) &= 0 \\ -mr\dot{\phi}^2 + k(r-l) &= 0 \\ (k - m\dot{\phi}^2)r &= kl \\ \therefore\ r_0 &= \frac{kl}{k - m\dot{\phi}^2} \end{aligned} \tag{18}$$

And the value of the corresponding pseudo-potential at equilibrium is

$$\text{“}V\text{”}_0 = \frac{mk(r_0^4 - lr_0^3)}{2mr_0^2} + \frac{k}{2}(r_0 - l)^2 = \frac{k}{2}(r_0^2 - lr_0) + \frac{k}{2}(r_0 - l)^2 \tag{19}$$

And

$$\frac{\partial^2\text{“}V\text{”}}{\partial r^2}|_{r_0} = \frac{3p_\phi^2}{mr_0^4} + k > 0 \tag{20}$$

The potential for the small displacement around r_0 is

$$\text{“}V\text{”}(r) = V_0 + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(r - r_0)^2 + ... \tag{21}$$

And the total energy is for $r \sim r_0$ is

$$E = \frac{m}{2}\dot{r}^2 + V_0 + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(r - r_0)^2 + ... \tag{22}$$

This shows that for small oscillations around the equilibrium position, we have an effective harmonic oscillator with frequency

$$\omega = \sqrt{\frac{\partial^2 V}{\partial r^2}/m} \tag{23}$$

In general, solving the radial part of the equation, in terms of "V" , we obtain

$$\begin{aligned}
\sqrt{\frac{2}{m}(E - \text{``}V\text{''})} &= \dot{r} \\
\frac{dr}{\sqrt{\frac{2}{m}(E - \text{``}V\text{''})}} &= dt \\
\int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2}{m}(E - \text{``}V\text{''})}} &= \int_{t_1}^{t_2} dt \\
\int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2}{m}(E - \text{``}V\text{''})}} &= t_2 - t_1
\end{aligned} \tag{24}$$