Chaos

Definition and Classification

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Contents

Terminology

Defining Chaos
- Sensitive Dependence
- Topological Transitivity
- Dense Periodic Orbits

Classification
- Equilibria
- Lyapunov Numbers and Exponents

Applications

References

Terminology

Map: a single valued "function" $f(x) : \mathbb{F} \to \mathbb{F}$ on some set $\mathbb{F}$.
Iteration: repeated application of a map i.e. $f^{(3)}(x) = f(f(f(x)))$.
Orbit: A sequence $S(x) = \{x, x_1, x_2, \cdots\} = \{x, f(x), f(f(x), \cdots\}$.
Fixed point: a value x_0 such that $f(x_0) = x_0$.
Periodic orbit: an orbit with $f^{(n)}(x_p) = x_p$ for any integer $n \geq 0$.

Definition

A map is deemed “chaotic” if it has the following properties:

- Sensitive dependence on initial conditions,
- Topological transitivity,
- Dense periodic orbits.

Similarly, a chaotic orbit is not periodic, stationary, or divergent. Chaos is a property of orbits *typical* of systems with nonlinear dynamics.

Sensitive Dependence

Definition

For any orbit of any map (eg $S(x)$ for $f(x) : \mathbb{C} \to \mathbb{C}$ where $\mathbb{C}$ is the set of all complex numbers), sensitive dependence on initial conditions means that for $|x_1 - x_2| < \epsilon$ for ϵ as small as we like, $|f^{(n)}(x_1) - f^{(n)}(x_2)| > \delta$ for any $\delta > 0$ we choose.

In the words of Edward Lorenz, "Chaos: When the present determines the future, but the approximate present does not approximately determine the future."

Sensitive Dependence

Example

A very simple example of a map with sensitive dependence on initial conditions is a doubling map: $f(x) = 2x$. Under a single iteration of this map, the value x doubles.
Notice that if we choose x_1 and x_2 such that $x_1 - x_2 < \epsilon$ for any $\epsilon > 0$ then $f(x_1) - f(x_2) < 2\epsilon$ for the same ϵ. Thus, as we continue to iterate, we see that $f^{(n)}(\theta_1) - f^{(n)}(\theta_2) < 2n\epsilon$ and thus our results may be arbitrarily far apart.

All this to say the only to predict similar results over an unknown number of iterations of the doubling map is to use an identical initial value.

Sensitive Dependence

Doubling Map Figure

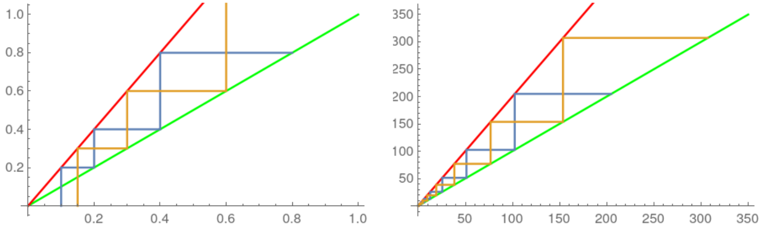

Figure 1: Double Steps

These figures show what happens when we repeatedly double similar values. The red line is $y = 2x$ and the green line is $y = x$. The steps indicate what happens when we iterate the map. We observe that though we begin with $|x_1 - x_2| = 0.05$, we have $|f^{(20)}(x_1) - f^{(20)}(x_2)| \approx 50$.

Topological Transitivity

Definition

A map $f(z) : \mathbb{C} \rightarrow \mathbb{C}$ is said to be **topologically transitive** if for any two nonempty sets A and B, there is some integer n such that for the n^{th} iterate of f, $f^{(n)}(A) \cap B \neq 0$.
In other words, any value plugged into a topologically transitive map may produce any other value (at all) if the map is iterated enough times.

Topological Transitivity

Example

The logistic map $x_{n+1} = rx_n(1 - x_n)$ is a well known example of a map which may exhibit chaos. Note that r is a parameter which largely determines the behavior of the recurrence. Notice that the function $f(x) = rx(1 - x) = rx - rx^2$, $x \in [0, 1]$ is a concave down parabola that intersects the x axis at $x = 0$ and $x = 1$ for any value of r. Therefore the maximum value of the function is taken at $x = 1/2 \implies \max(f(x) : x \in [0, 1]) = r/2 - r/4$. If we set $r = 4$ then, $\max(f(x)) = 1$ and $f(x)$ may take any value in its domain. This means there is at least one point $x \in [0, 1]$ which may take any desired value between 0 and 1 after some number of iterations of $f(x)$.

Topological Transitivity

Logistic Map Bifurcations

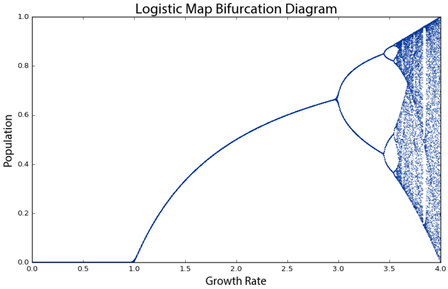

Figure 2: Logistic Map

The above plot shows where the periodic points for the map $x_{n+1} = rx_n(1 - x_n)$ are. $0 < r < 4$ is on the horizontal axis, and $0 < x < 1$ is on the vertical axis. For $0 < r <\sim 3$ there is only a fixed point, but for $\sim 3 < r <\sim 3.5$ there is a period 2 orbit.

Dense Periodic Orbits

Definition

A periodic orbit with period n is defined by the property

$$x_R = f^{(n)}(x_R).$$

Density of periodic orbits in a chaotic system is a bit hard to understand, although it is really the most essential piece of the definition. Essentially, no matter what starting value is chosen (x, for $f^{(n)}(x)$), the distance between x and a point on some periodic orbit is arbitrarily small: that is for any $\epsilon > 0$, $|x - x_R| < \epsilon$ for some x_R.

Intuitively, this is weird. It means that no matter what the eventual behavior of the trajectory you choose is, there is a point infinitessimally far away from you starting point that is on a repeating trajectory.

Dense Periodic Orbits

Explanation

First, we'll parse the word 'dense.' Here we mean dense in the same way rational numbers are dense in the real numbers; there are always at least two (one above and one below) rational numbers within any radius of any other number.
What this means in the context of periodic orbits is that no matter what trajectory you're interested in with respect to a chaotic system, it's always at least almost a periodic trajectory. The close periodic trajectories may have huge periods like $n = 238974602$ (in the same way a nearby rational number could be 23679065432/124764 for example).

Classification

Equilibria

Note that Equilibria in this sense are *orbits*, not points.
Chaotic maps have a few classes of equilibria; much like planetary orbits they may be stable or unstable, or metastable. We categorize them the following way:

- **Attractor**: a stable orbit which nearby unstable orbits tend to approach,
- **Source**: an unstable orbit from which small deviations result in wildly different trajectories (like a ball balanced atop a cone)

Note that both attractors and sources may be chaotic in nature. In fact there is a class of attractors called "Strange Attractors" (eg Lorenz attractor, Hénon map) which have fractal (self repeating) structure and are often chaotic.

Classification

Equilibria: Stability

Much like optimization of functions, the stability of any orbit S of a map f is determined by the first derivative of the map over the orbit. That is to say

- if $(f^{(n)})'(x_1) < 1$ then $S(x_1) = f^{(n)}(x_1)$ is an attractive orbit:
- if $(f^{(n)})'(x_1) > 1$ then $S(x_1) = f^{(n)}(x_1)$ is a source.

Classification

Lorenz Attractor

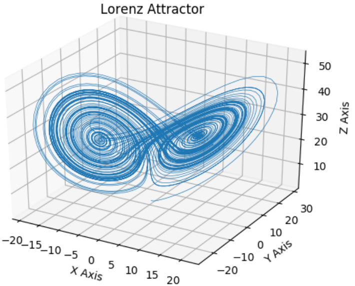

Figure 3: Lorenz Attractor

The Lorenz attractor is a chaotic orbit with $(f^{(n)})'(x) < 1$.

Lyapunov numbers and exponents are a measure of the stability of an orbit. Following convention, the Lyapunov exponent $\lambda = \ln L$ where L is the Lyapunov number.

By definition, for any point on any orbit $\{x_1, x_2, x_3, \cdots\}$, the Lyapunov number is given by:

$$L(x_1) = \lim_{n\to\infty} (|f'(x_1)| \cdots |f'(x_n)|)^{1/n}.$$

So the Lyapunov Exponent is:

$$\lambda(x_1) = \lim_{n\to\infty} \frac{1}{n} \ln(|f'(x_1)| \cdots |f'(x_n)|)$$

Lyapunov numbers give us the simplest method of determining whether an orbit is chaotic (aperiodic) or not:

any orbit S with $L(x) > 1$ for any $x \in S$ is chaotic.

We use the qualifier "aperiodic" here because for $n < \infty$ (the feasible case for computing Lyapunov numbers) we miss the possible limit behavior $\lim_{n\to\infty} f^{(n)}(x_0) = x_0$.

Chaos Theory in Everyday Life

Since almost every dynamical system has chaos in some regime, the mathematical study of chaos is applied to increase our understanding. Some examples include:

- weather patterns,
- traffic patterns,
- reproduction in biological systems,

and many others. Being able to classify and "predict" behavior (eg by finding attractive orbits close to the current behavior) there is much to gain.

Chaos theory in Physics

Per slide 4, chaos is typical of nonlinear systems. Thus knowledge of the behavior of chaotic orbits is useful in many classical and quantum systems.

In classical mechanics, some examples are

Damped Driven Harmonic Osicllator	Celestial Mechanics	Coupled Oscillators

In quantum mechanics, any system with more than one particle has associated chaos. A 'simple' quantum system with well studied chaotic effects is two particle spin coupling.

References

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Figure 2: https://www.researchgate.net/publication/306226253_Visual_Analysis_of_Nonlinear_Dynamical_Systems_Chaos_Fractals_Self-Similarity_and_the_Limits_of_Prediction/figures?lo=1

Figure 3: https://matplotlib.org/3.1.0/gallery/mplot3d/lorenz_attractor.html