Chaos

Definition and Classification

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Contents

Terminology

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- Fixed point: $x_0 \implies f(x_0) = x_0$.
- Periodic orbit: an orbit with $f^{(n)}(x_p) = x_p$ for $n \geq 0$.

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Similarly, a chaotic orbit is not periodic, stationary, or divergent.

Chaos is a property of orbits *typical* of systems with nonlinear dynamics.

Sensitive Dependence

Definition

For any orbit of any map sensitive dependence on initial conditions means that for $|x_1 - x_2| < \epsilon$ for ϵ as small as we like, $|f^{(n)}(x_1) - f^{(n)}(x_2)| > \delta$ for any $\delta > 0$ we choose.

In the words of Edward Lorenz, “Chaos: When the present determines the future, but the approximate present does not approximately determine the future.”

Sensitive Dependence

Example

A very simple example of a map with sensitive dependence on initial conditions is a doubling map: $f(x) = 2x$.

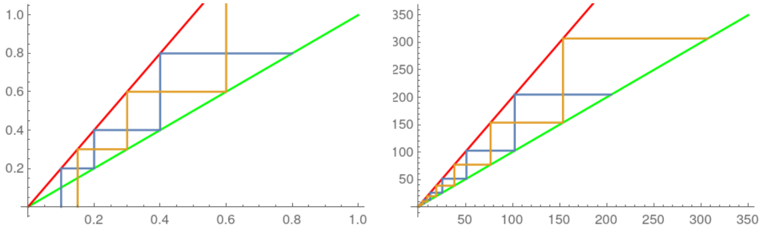

Figure 1: Double Steps

Topological Transitivity

Definition

A map is **topologically transitive** if for any two nonempty sets A and B, there is some integer n such that $f^{(n)}(A) \cap B \neq 0$.
In other words, any value plugged into a topologically transitive map may produce any other value (at all) if the map is iterated enough times.

Topological Transitivity

Example

The logistic map $x_{n+1} = rx_n(1 - x_n)$:

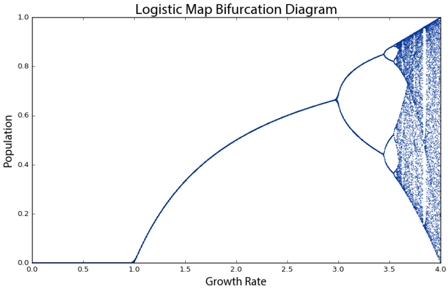

Figure 2: Logistic Map

This is a bifurcation plot. $0 < r < 4$ is on the horizontal axis, and $0 < x < 1$ is on the vertical axis. For $0 < r <\sim 3$ there is only a fixed point, but for $\sim 3 < r <\sim 3.5$ there is a period 2 orbit.

Dense Periodic Orbits

Definition

Density of periodic orbits means that no matter what starting value is chosen (x, for $f^{(n)}(x)$), the distance between x and a point on some periodic orbit is arbitrarily small: that is for any $\epsilon > 0$, $|x - x_R| < \epsilon$ for some x_R.

Classification

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- if $(f^{(n)})'(x_1) < 1$ then $S(x_1) = f^{(n)}(x_1)$ is an attractive orbit:
- if $(f^{(n)})'(x_1) > 1$ then $S(x_1) = f^{(n)}(x_1)$ is a source.

Classification

Lorenz Attractor

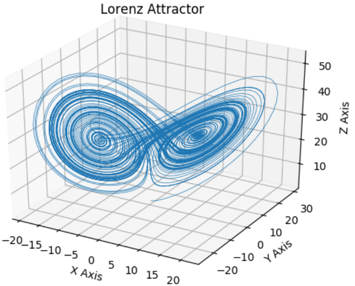

Figure 3: Lorenz Attractor

The Lorenz attractor is a chaotic orbit with $(f^{(n)})'(x) < 1$.

Lyapunov numbers and exponents are a measure of the stability of an orbit. The Lyapunov exponent $\lambda = \ln L$ where L is the Lyapunov number.

For any point on any orbit $\{x_1, x_2, x_3, \cdots\}$, the Lyapunov number is given by:

$$L(x_1) = \lim_{n\to\infty} (|f'(x_1)| \cdots |f'(x_n)|)^{1/n}.$$

So the Lyapunov Exponent is:

$$\lambda(x_1) = \lim_{n\to\infty} \frac{1}{n} \ln(|f'(x_1)| \cdots |f'(x_n)|)$$

References

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Figure 2: https://www.researchgate.net/publication/306226253_Visual_Analysis_of_Nonlinear_Dynamical_Systems_Chaos_Fractals_Self-Similarity_and_the_Limits_of_Prediction/figures?lo=1

Figure 3: https://matplotlib.org/3.1.0/gallery/mplot3d/lorenz_attractor.html