Classical Mechanics - Lecture notes 4/23/20

This derivation addresses several things we've learned over the last 2-3 weeks: Lagrangian formalism for continuous degrees of freedom, electromagnetic interaction in Lagrangian formulation, and relativity.

Consider a system which has a pre-determined, given charge density and current density distribution, expressed by the 4 components of the charge current-density four-vector $j^\mu(x^\mu) = \left(c\rho, \vec{j}\right)(ct, \vec{x})$ everywhere in space and time. We consider the 4 components of the 4-vector electromagnetic potential $A^\mu(x^\mu) = \left(\frac{\Phi}{c}, \vec{A}\right)(ct, \vec{x})$ as the 4 continuous, independent degrees of freedom, all dependent on space-time. (NOTE: In the following we assume the Lorentz gauge, i.e. $\partial_\mu A^\mu = 0$).

In the following, we also use the electromagnetic field (2-)tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$ where $\partial^\mu = g^{\mu\alpha}\partial_\alpha$ and

$\partial_\alpha = \left(\frac{\partial}{\partial ct}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the usual 4-dimensional gradient. In other words, $F^{\mu\nu}$ can be considered a function of the space-time **derivatives** of our continuous degrees of freedom. Following our formalism, we can now define a Lagrangian density $\ell(A^\mu, \partial_\nu A^\mu) = -j_\mu A^\mu - \frac{1}{4\mu_o} F^{\mu\nu} F_{\mu\nu}$. Here, the first part is the "potential energy" due to the interaction between the current and the field (analog to the term $q\vec{v} \cdot \vec{A} - q\Phi$ in the Lagrangian for a single charge interacting with a given electromagnetic field) and the second part is just equal to the negative of the electromagnetic field energy density, $w = \frac{\epsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2$.

First, we need to re-express the Lagrangian density in terms of only the fields A^μ and their derivatives $\partial_\nu A^\mu$:

$\ell(A^\mu, \partial_\nu A^\mu) = -j_\mu A^\mu - \frac{1}{4\mu_o} F^{\mu\nu} F_{\mu\nu} = -j_\mu A^\mu - \frac{1}{4\mu_o}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$. We use the metric tensor to raise the indices of A and lower those of ∂ as needed:

$$\ell(A^\mu, \partial_\nu A^\mu) = -j_\mu A^\mu - \frac{1}{4\mu_o}\left(g^{\mu\rho}\partial_\rho A^\nu - g^{\nu\rho}\partial_\rho A^\mu\right)\left(g_{\nu\sigma}\partial_\mu A^\sigma - g_{\mu\sigma}\partial_\nu A^\sigma\right).$$

Straightforward multiplication of the last two brackets gives

$$\begin{aligned}
&\left(g^{\mu\rho}g_{\nu\sigma}\partial_\rho A^\nu \partial_\mu A^\sigma - g^{\mu\rho}g_{\mu\sigma}\partial_\rho A^\nu \partial_\nu A^\sigma - g^{\nu\rho}g_{\nu\sigma}\partial_\rho A^\mu \partial_\mu A^\sigma + g^{\nu\rho}g_{\mu\sigma}\partial_\rho A^\mu \partial_\nu A^\sigma\right) \\
&= \left(g^{\mu\rho}g_{\nu\sigma}\partial_\rho A^\nu \partial_\mu A^\sigma - g^\rho_\sigma \partial_\rho A^\nu \partial_\nu A^\sigma - g^\rho_\sigma \partial_\rho A^\mu \partial_\mu A^\sigma + g^{\nu\rho}g_{\mu\sigma}\partial_\rho A^\mu \partial_\nu A^\sigma\right) \\
&= \left(g^{\mu\rho}g_{\nu\sigma}\partial_\rho A^\nu \partial_\mu A^\sigma - \partial_\rho A^\nu \partial_\nu A^\rho - \partial_\rho A^\mu \partial_\mu A^\rho + g^{\nu\rho}g_{\mu\sigma}\partial_\rho A^\mu \partial_\nu A^\sigma\right) \\
&= \left(2g^{\mu\rho}g_{\nu\sigma}\partial_\rho A^\nu \partial_\mu A^\sigma - 2\partial_\rho A^\nu \partial_\nu A^\rho\right)
\end{aligned}$$

where we made use of the fact that the metric tensor is symmetric and its own inverse: $g^{\mu\rho}g_{\mu\sigma} = g^{\rho\mu}g_{\mu\sigma} = \delta_{\rho\sigma}$. (The last line also uses the fact that one can rename any indices over which we are summing.) We are now ready to calculate the various derivatives required by Eq. (13.23) in Goldstein; to avoid any confusion, I will be using brand new indices α, β:

$\dfrac{\partial \ell}{\partial A^\alpha} = -j_\alpha$ and

$$\frac{\partial \ell}{\partial(\partial_\beta A^\alpha)} = -\frac{1}{4\mu_o}\left(2g^{\mu\beta}g_{\alpha\sigma}\partial_\mu A^\sigma + 2g^{\beta\rho}g_{\nu\alpha}\partial_\rho A^\nu - 2\partial_\alpha A^\beta - 2\partial_\alpha A^\beta\right) = \frac{-g^{\beta\rho}g_{\alpha\nu}\partial_\rho A^\nu + \partial_\alpha A^\beta}{\mu_o}$$

where we have again used the fact that the metric tensor is symmetric and that we can rename any indices over which we are summing. Now we are ready to write down the full equation of motion for the component A^α:

$$\partial_\beta \frac{\partial \ell}{\partial(\partial_\beta A^\alpha)} - \frac{\partial \ell}{\partial A^\alpha} = \frac{1}{\mu_o}\left(-g^{\beta\rho}g_{\alpha\nu}\partial_\beta \partial_\rho A^\nu + \partial_\beta \partial_\alpha A^\beta\right) + j_\alpha = 0$$

$$= \frac{1}{\mu_o}\left(-g_{\alpha\nu}\partial^\rho \partial_\rho A^\nu + \partial_\alpha \partial_\beta A^\beta\right) + j_\alpha \Rightarrow \quad g_{\alpha\nu}\partial^\rho \partial_\rho A^\nu = \partial^\rho \partial_\rho A_\alpha = \mu_o j_\alpha$$

Here, we used the Lorentz-gauge condition (together with the fact that derivatives can be interchanged) to cancel the second term in the bottom row.

The final result relates two 1-forms to each other; we can easily apply the inverse of $g_{\alpha\nu}$ on both sides and end up with $\partial^\rho \partial_\rho A^\nu = \mu_o j^\nu$. But this is equivalent to Maxwell's equations for the 4-vector potential, see Eq. (7.67a) on page 297 in Goldstein! So we were able to derive Maxwell's equations for the electromagnetic field from our Lagrangian density.

Here is a screenshot from today's whiteboard:

$$\cdot W = \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2 = \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} \qquad (\partial^\mu A^\nu - \partial^\nu A^\mu) \qquad \frac{\vec{E}^2}{c^2} \qquad c^2\mu_0 = \frac{1}{\varepsilon_0}$$

$$\mathcal{L} = -j_\nu A^\nu - \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} = \mathcal{L}\left(A^\nu(x^\mu),\ \partial_\nu A^\nu(x^\mu)\right) \qquad \partial_\nu = \frac{\partial}{\partial x^\nu}$$

$$= -j_\nu A^\nu - \underbrace{\frac{1}{4\mu_0}\left(g^{\mu\rho}\partial_\rho A^\nu - g^{\nu\rho}\partial_\rho A^\mu\right)\left(\partial_\mu g_{\nu\sigma}A^\sigma - \partial_\nu g_{\mu\sigma}A^\sigma\right)}$$

$$\frac{1}{4\mu_0}\Big(g^{\mu\rho}g_{\nu\sigma}\partial_\rho A^\nu \partial_\mu A^\sigma - \underbrace{\delta^\rho_\sigma\, \partial_\rho A^\nu \partial_\nu A^\sigma} - \delta^\rho_\sigma\, \partial_\rho A^\mu \partial_\mu A^\sigma + g^{\nu\rho}g_{\mu\sigma}\partial_\rho A^\mu \partial_\nu A^\sigma$$

$$\frac{1}{4\mu_0}\left(2\, g^{\mu\rho}g_{\nu\sigma}\partial_\rho A^\nu \partial_\mu A^\sigma - 2\partial_\rho A^\nu \partial_\nu A^\rho\right)$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\beta A^\alpha)} = -\frac{1}{4\mu_0}\left(2g^{\mu\beta}g_{\alpha\sigma}\,\partial_\mu A^\sigma\cdot 2 - 2\,\partial_\alpha A^\beta\cdot 2\right) \qquad \cancel{\partial_\beta A^\beta}$$

$$\partial_\beta \cdot \qquad = \frac{1}{\mu_0}\left(g^{\mu\beta}g_{\alpha\sigma}\,\partial_\beta\partial_\mu A^\sigma\right) \qquad \underbrace{\frac{\partial^2}{\partial ct^2} - \vec{\nabla}^2} \qquad =_c$$

$$= -\frac{1}{\mu_0}\partial^\mu\partial_\mu A_\alpha = \frac{\partial\mathcal{L}}{\partial A^\alpha} = -j_\alpha \Rightarrow \Box A_\alpha = \mu_0 j_\alpha$$