Classical Mechanics: PHYS 603
The Hamilton's Equations of Motion: Lecture Note Prepared by

Bashu Dev Khanal

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The Hamilton's Equations of Motion:
Consider q_i and $\dot{q}_i$ are generalized coordinates and velocities, then the Lagrangian is a function of q_i and $\dot{q}_i$ and t.
That is

$$L = L(q_i, \dot{q}_i, t)$$

The Lagrange equation of motion:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

where we can introduce

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{1}$$

and

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2}$$

The energy function is

$$h(q_i, \dot{q}_i, t) = \sum_{i=1}^{k} p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$$

or

$$dh(q_i,\dot{q}_i,t) = \sum_{i=1}^{k}(dp_i\dot{q}_i + p_i d\dot{q}_i) - \sum_{i=1}^{k}(\frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial q_i}dq_i) - \frac{\partial L}{\partial t}dt$$

$$= \sum_{i=1}^{k}(dp_i\dot{q}_i + p_i d\dot{q}_i) - \sum_{i=1}^{k}(p_i d\dot{q}_i + \frac{\partial L}{\partial q_i}dq_i) - \frac{\partial L}{\partial t}dt$$

$$dh(q_i,\dot{q}_i,t) = \sum_{i=1}^{k} dp_i\dot{q}_i - \sum_{i=1}^{k} \dot{p}_i dq_i - \frac{\partial L}{\partial t}dt \tag{3}$$

Since,

$$H(q_i,p_i,t) = h[q_i,\dot{q}_i(p_j,q_j,t),t]$$

Then,

$$\frac{\partial H}{\partial p_i} = \frac{\partial h}{\partial p_i} = \dot{q}_i$$

and

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i$$

Hence,

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{4}$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{5}$$

and

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{6}$$

Equations (4) and (5) are known as Hamilton's canonical equations of motion. These equations are first order partial differential equations replacing the n second-order Lagrange's equations of motion.

In the large classes of cases:
The Lagrangian can be written as,

$$\mathcal{L} = \frac{1}{2}\dot{\vec{q}}^T\Pi\dot{\vec{q}} + \dot{\vec{q}}^T.\vec{a} + \mathcal{L}_0(q_i,t)$$

Then,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (\Pi\dot{\vec{q}})_i + (\vec{a})_i$$

$$\implies p_i = (\Pi\dot{\vec{q}})_i + (\vec{a})_i$$

$$\implies \dot{q}_i = \Pi^{-1}(\vec{p}_i - \vec{a})$$

$$\implies \sum_i p_i\dot{q}_i = \dot{\vec{q}}^T\Pi\dot{\vec{q}} + \dot{\vec{q}}^T.\vec{a}$$

Then the energy function is

$$h = \sum_i p_i\dot{q}_i - \mathcal{L}$$

$$= \dot{\vec{q}}^T\Pi\dot{\vec{q}} + \dot{\vec{q}}^T.\vec{a} - \{\frac{1}{2}\dot{\vec{q}}^T\Pi\dot{\vec{q}} + \dot{\vec{q}}^T.\vec{a} + \mathcal{L}_0(q_i,t)\}$$

Hence,

$$\boxed{h = \frac{1}{2}\dot{\vec{q}}^T\Pi\dot{\vec{q}} - \mathcal{L}_0(q_i,t)\}}$$

Now, the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}(\vec{p}-\vec{a})^T\Pi^{-1T}(\vec{p}-\vec{a}) - \mathcal{L}_0(\vec{q},t)$$

Since, $\Pi^{-1T} = \Pi^{-1}$

Hence, the Hamiltonian in the large classes of cases:

$$\boxed{\mathcal{H} = \frac{1}{2}(\vec{p}-\vec{a})^T\Pi^{-1}(\vec{p}-\vec{a}) - \mathcal{L}_0(\vec{q},t)}$$

Example 1.
Consider a particle of mass M is falling from a certain height under the influence of gravitational force only. (a) Find the Lagrange's and Hamilton's equations of motion. (b) Solve the equation of motion to find the position and momentum of the particle at any time t, using initial condition that at $t = 0$, $y(0) = y_0$ and $\dot{y}(0) = v_{0y}$.

Solution:

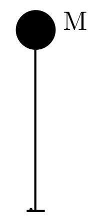

Fig. Vertical motion of a particle

The kinetic energy of the particle is

$$T = \frac{1}{2}M\dot{y}^2$$

The potential energy of the particle is

$$V = Mgy$$

Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}M\dot{y}^2 - Mgy$$

The Lagrange equation of motion is

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) - \frac{\partial L}{\partial y} = 0$$

or

$$\frac{d}{dt}(M\dot{y}) - (-Mg) = 0$$

Hence,

$$\boxed{M\ddot{y} = -Mg}$$

is the required Lagrange's equation of motion.
Again,
The Hamiltonian is givn by

$$H(y, p_y) = \frac{1}{2}M\dot{y}^2 + Mgy$$

$$= \frac{p_y^2}{2M} + Mgy$$

Then, the Hamilton's equations of motion are given by

$$\boxed{\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{M}}$$

and

$$\boxed{\dot{p}_y = -\frac{\partial H}{\partial y} = -Mg}$$

(b)
To calculate the position and momentum of the particle at any time t:
We have,

$$p_y = M\dot{y}$$

or,

$$\dot{p}_y = M\ddot{y}$$

or,

$$-Mg = M\ddot{y}$$

or,

$$\ddot{y} = -g$$

or,

$$\dot{y} = -gt + c$$

or,

$$\frac{dy}{dt} = -gt + c$$

or,

$$\boxed{y(t) = -\frac{1}{2}gt^2 + ct + c_1} \tag{7}$$

To calculate constants c and c_1:
At $t = 0$, $y(0) = y_0 = c_1 \implies c_1 = y_0$ and at $t = 0$, $\dot{y}(0) = v_{0y} = c \implies c = v_{0y}$ Putting the value of c and c_1 in equation (7):

$$\boxed{y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2}$$

which gives the position of the particle at any time t.
For the momentum:
Here,

$$\dot{y}(t) = v_{oy} - gt$$

Then,

$$\boxed{p_y(t) = M\dot{y}(t) = M(v_{0y} - gt)}$$

is the momentum of the particle at any time t.
Example 2.
Consider a particle is in electromagnetic field with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^2 - q\Phi + q\vec{A}.\dot{\vec{r}}$$

(a) Find the canonical momentum.
(b) Find the Hamiltonian and Hamilton's equations of motion.

Solution:
(a)
Since,

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^2 - q\Phi + q\vec{A}.\dot{\vec{r}}$$

The canonical momentum is given by

$$\vec{p} = \frac{\partial L}{\partial \dot{r}} = m\dot{\vec{r}} + q\vec{A}$$

Hence, the canonical momentum is

$$\boxed{\vec{p} = m\dot{\vec{r}} + q\vec{A}} \tag{8}$$

(b)
The Hamiltonian is given by

$$\mathcal{H} = \vec{p}.\dot{\vec{r}} - \mathcal{L}(r, \dot{r}, t)$$

$$= (m\dot{\vec{r}} + q\vec{A}).\dot{\vec{r}} - (\frac{1}{2}m\dot{\vec{r}}^2 - q\Phi + q\vec{A}.\dot{\vec{r}})$$

$$= \frac{1}{2}m\dot{\vec{r}}^2 + q\Phi$$

Putting the value of $\dot{\vec{r}}$ from equation (8):

$$\boxed{\mathcal{H}(\vec{r}, \vec{p}) = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\Phi}$$

Hence, the required Hamilton's equations of motion are

$$\boxed{\dot{\vec{r}} = \frac{\partial \mathcal{H}}{\partial p} = \frac{\vec{p} - q\vec{A}}{m}}$$

and,

$$\boxed{\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial r} = q(\frac{\vec{p} - q\vec{A}}{m})\nabla\vec{A} - q\nabla\Phi}$$