Constraints, Generalized Coordinates and Euler Langrange's Equation

Randika Dodangodage

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1 Constraints

A particle or a system of particles is described by a set of coordinates. There are various forces acting on the particles which should to be taken to account in order to understand the mechanics. In most of the cases the motion of the system is limited to some conditions which are known as **constraints**.

Example 1:
Consider a motion of a particle in $x - y$ plane which is tied from a string to a point.The particle will move along a circular path around the fixed point. The length of the string limits the motion which is a constraint.

1.1 Holonomic constraints

If the constraint depends only on the position at time t and can be written in the form

$$F_k(r_i, t) = 0$$

$$i = 1, 2, ..., 3N$$
$$k = 1, 2, ..., m$$
$$N = \text{No: of particles}$$
$$m = \text{No: of constraints}$$

then F_k is **Holonomic**.

1.2 Non Holonomic constraints

If it is in the following forms it is **Non Holonomic**.

$$F_k(r_i, t) < 0$$
$$F_k(r_i, t) > 0$$
$$F_k(r_i, \dot{r}_i, ...)$$

2 Generalized coordinates (q_i)

When a system has constraints the coordinates are no longer independent. They depend on constraints. Therefore the concept generalized coordinates is introduced.
A system of N particles has k number of constraints $3N$ number of coordinates and $3N-k$ number of degrees of freedom.

Consider the example 1

Cartesian coordinates for this system are x, y. We can represent this system with generalized coordinates which includes the constraint

$$q_1 = \sqrt{x^2+y^2} = r$$
$$q_2 = \tan^{-1}\left(\frac{y}{x}\right) = \phi$$

$$x = r\cos\phi = q_1 \cos q_2$$
$$y = r\sin\phi = q_1 \sin q_2$$

3 Generalized Force (Q_j)

Consider the equation of motion for a system in equilibrium.

$$m_i\ddot{r}_i - F_i = 0$$

Now introduce a virtual displacement δr_i fulfilling the constraint.Then we can write

$$\sum_i \big(m_i\ddot{r}_i - F_i\big)\delta r_i = 0$$

We can split the total force (F_i) in to two forces as forces of constraints (F_i^c) and all other forces (f_i^o).

$$F_i = F_i^c + f_i^o$$
$$\sum_i \big(m_i\ddot{r}_i - F_i^c - f_i^o\big)\delta r_i = 0$$

F_i^c cannot do work. ($\sum_i F_i^c \delta r_i = 0$). The virtual displacement δr_i is connected to virtual displacement δq_i by

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j}\delta q_j$$

$$\sum_i m_i \frac{d\dot{r}_i}{dt}\delta r_i = \sum_i f_i^o \delta r_i$$
$$\sum_{i,j} m_i \frac{d\dot{r}_i}{dt}\frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{i,j} f_i^o \frac{\partial r_i}{\partial q_j}\delta q_j$$

Define Q_j as

$$Q_j = \sum_i \frac{\partial r_i}{\partial q_j} f_i^o$$

$$\sum_{i,j} m_i \frac{d\dot{r}_i}{dt} \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j \tag{1}$$

Now consider

$$\begin{aligned}
\sum_i m_i \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j}\right) &= \sum_i m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) + \sum_i m_i \frac{d\dot{r}_i}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) \\
&= \sum_i m_i \dot{r}_i \left(\frac{\partial \dot{r}_i}{\partial q_j}\right) + \sum_i m_i \frac{d\dot{r}_i}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) \\
\sum_i m_i \frac{d\dot{r}_i}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) &= \sum_i m_i \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j}\right) - \sum_i m_i \dot{r}_i \left(\frac{\partial \dot{r}_i}{\partial q_j}\right)
\end{aligned}$$

Note

$$\begin{aligned}
T &= \frac{1}{2}\sum_i m_i \dot{r}_i^2 \\
\frac{\partial T}{\partial q_j} &= \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_j} \\
\frac{\partial T}{\partial \dot{q}_j} &= \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j}
\end{aligned}$$

and $\frac{\partial r_i}{\partial q_j} = \frac{\partial \dot{r}_i}{\partial \dot{q}_j}$

$$\begin{aligned}
\sum_i m_i \frac{d\dot{r}_i}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) &= \sum_i m_i \frac{d}{dt}\left(\dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) - \sum_i m_i \dot{r}_i \left(\frac{\partial \dot{r}_i}{\partial q_j}\right) \\
&= \frac{d}{dt}\sum_i m_i \left(\dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) - \sum_i m_i \dot{r}_i \left(\frac{\partial \dot{r}_i}{\partial q_j}\right) \\
&= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}
\end{aligned}$$

Substituting this in to equation (1)

$$\begin{aligned}
\sum_j \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right)\delta q_j &= \sum_j Q_j \delta q_j \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} &= Q_j
\end{aligned}$$

4 Lagrange's Equation

We can derive the forces from a scalar potential (V) as follows.

$$\begin{aligned}
F_i = -\nabla V &= -\frac{\partial V}{\partial r_i} \\
Q_j &= \sum_i \frac{\partial r_i}{\partial q_j} F_i \\
&= -\sum_i \frac{\partial r_i}{\partial q_j} \frac{\partial V}{\partial r_i} \\
Q_j &= -\frac{\partial V}{\partial q_j}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} &= Q_j \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} &= -\frac{\partial V}{\partial q_j} \\
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}\big(T - V\big) &= 0
\end{aligned}$$

The potential V doesn't depend on the velocities $\dot{q}_j$

$$\frac{d}{dt}\frac{\partial (T - V)}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}\big(T - V\big) = 0$$

Define $\mathcal{L} = T - V$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$