3/28/06

From 3/23:

$$F_2 = \sum q_i P_i + dt\, H(\vec{q}, \vec{P}, t) \Rightarrow$$

$$\vec{\eta}(t_0) \rightarrow \vec{\zeta} = \vec{\eta}(t + dt)$$

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i + \dot{q}_i\, dt$$

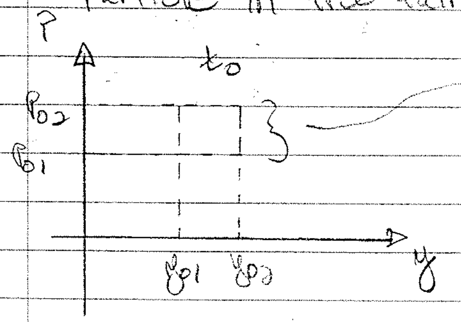
$$p_i = \frac{\partial F_2}{\partial q_i} = P_i - \dot{p}_i\, dt$$

$$P_i = p_i + \dot{p}_i\, dt$$

$$\vec{\eta} \rightarrow \vec{\zeta} \Rightarrow \bar{M}_{ij} = \frac{\partial \vec{\zeta}_i}{\partial \vec{\eta}_j} \quad \text{so} \quad \bar{M}\bar{\bar{J}}\bar{M}^T = \bar{\bar{J}}, \quad \bar{M}^T\bar{\bar{J}}\bar{M} = \bar{\bar{J}}$$

<u>Example</u>: Louisville's Theorem

Particle in free fall



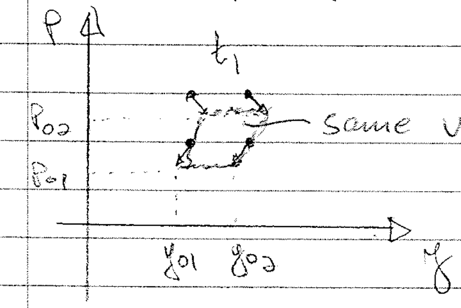
Rectangle bounds systems with different initial conditions but same Hamiltonian $\Rightarrow$

$$H = \frac{p^2}{2m} + mgy$$

where $y(t) = y_0 + \frac{p_0}{m} t - \frac{1}{2} g t^2$

and $p(t) = p_0 - mgt$

Consider the four corners of the rectangle at a later time



same volume, different shape

Here the professor explained the secrets of the Universe, but if you don't come to class, you'll never know them. Hint: The answer is "49"

Poisson Brackets

Consider two functions $u(\vec{p}, \vec{q}, t)$, $v(\vec{p}, \vec{q}, t)$.
The Poisson Bracket is defined as

$$[u, v]_{\vec{q}, \vec{p}} = \sum_i \left(\partial u/\partial q_i \, \partial v/\partial p_i - \partial u/\partial p_i \, \partial v/\partial q_i \right)$$

Some examples:

$$[q_j, q_k]_{\vec{q}, \vec{p}} = 0$$

$$[p_j, p_k]_{\vec{q}, \vec{p}} = 0$$

$$[q_j, p_k]_{\vec{q}, \vec{p}} = \delta_{jk}$$

$$[u, v] = -[v, u]$$

Notice $\sum_i \left(\partial u/\partial q_i \, \partial v/\partial p_i - \partial u/\partial p_i \, \partial v/\partial q_i \right) = \left(\frac{\partial u}{\partial \vec{\eta}} \right)^T \bar{\bar{J}} \left(\frac{\partial v}{\partial \vec{\eta}} \right)$

so $\quad [\eta_j, \eta_k]_{\vec{\eta}} = \bar{\bar{J}}_{jk}$

Suppose $\vec{q}, \vec{p}$ are canonically transformed. Then

$$[\xi_j, \xi_k]_{\vec{\eta}} = \left(\frac{\partial \xi_j}{\partial \vec{\eta}} \right)^T \bar{\bar{J}} \left(\frac{\partial \xi_k}{\partial \vec{\eta}} \right)$$

To find a component,

$$[\vec{\xi}, \vec{\xi}]_{\vec{\eta}} = \left(\frac{\partial \vec{\xi}}{\partial \vec{\eta}} \right)^T \bar{\bar{J}} \left(\frac{\partial \vec{\xi}}{\partial \vec{\eta}} \right) = \bar{\bar{M}}^T \bar{\bar{J}} \bar{\bar{M}} = \bar{\bar{J}}$$

which implies $[\vec{\xi}, \vec{\xi}]_{\vec{\eta}} = [\vec{\xi}, \vec{\xi}]_{\vec{\xi}}$. Vice versa,

$$[\vec{\eta}, \vec{\eta}]_{\vec{\xi}} = [\vec{\eta}, \vec{\eta}]_{\vec{\eta}}$$

$\Rightarrow$ Poisson brackets are independent of canonical variables

[invariant under canonical transformations]

Example 1

Let $u = u(\vec{p}, \vec{q}, t)$. Then

$$[u, H] = \sum_i \left(\partial u/\partial q_i \; \partial H/\partial p_i - \partial u/\partial p_i \; \partial H/\partial q_i \right)$$

$$= \sum_i \left(\partial u/\partial q_i \; \dot{q}_i + \partial u/\partial p_i \; \dot{p}_i \right)$$

$$= \frac{du}{dt} - \frac{\partial u}{\partial t}$$

So

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

If u is conserved, then $\cancel{[u, H] = 0} \Rightarrow [u, H] + \frac{\partial u}{\partial t} = 0$

$$[H, u] = \partial u/\partial t$$

Observe

$$\left.\begin{array}{l} [\vec{p}, H] = \dot{\vec{p}} \\ [\vec{q}, H] = \dot{\vec{q}} \end{array}\right\} \quad [\vec{\eta}, H] = \dot{\vec{\eta}} = \underbrace{\left(\frac{\partial \vec{\eta}}{\partial \vec{\eta}}\right)}_{= \mathbb{1}} (J) \left(\frac{\partial H}{\partial \vec{\eta}}\right)$$

Example 2

$$[u, p_j] = \frac{\partial u}{\partial q_j} \Rightarrow$$

$$[u, p_j]\, \delta q_j = \frac{\partial u}{\partial q_j}\, \delta q_j = \partial u$$

$$dt\,[u, H] = \left(\frac{du}{dt} - \frac{\partial u}{\partial t}\right) \delta t$$

If $\frac{\partial u}{\partial t} = 0$, then

$$dt\,[u, H] = du$$

— OR —

$$\vec{\eta} + \delta\vec{\eta} \; ; \quad \delta\vec{\eta} = dt\,[\vec{\eta}, H] = dt\, J \left(\frac{\partial H}{\partial \vec{\eta}}\right)$$