Hamilton's Principle with (explicit) constraints

Earlier we defined Hamilton's principle of least action in a system where the constraints are implemented implicitly through the choice of a reduced set of independent q_i that "automatically" fulfill the constraints:

$$\int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)\, dt = \ Extremum$$

Then the Euler Lagrange equation that corresponds to the integral becomes the Lagrange's equation of motion. $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$; i = 1,2,3,............ n

To study Hamilton's principle for systems with variables that are connected by equations of constraint (not independent), forces of constraints should be considered while deriving Lagrange's equations. Use of the variational principle is possible for these systems if equations of constraints can be written as, $g_k(q_1, q_2, \dots\dots, q_N) = 0$. Now the action equation will be in the form

$$\int_{t_1}^{t_2} [\mathcal{L} + \sum_k \lambda_k g_k]\, dt = extremum$$

where $\lambda_k(t)$ are the Lagrange multipliers and $g_k(q_i) = 0$ are the equations of constraint. So, incorporating Hamilton's principle with constraints we can write a new equation,

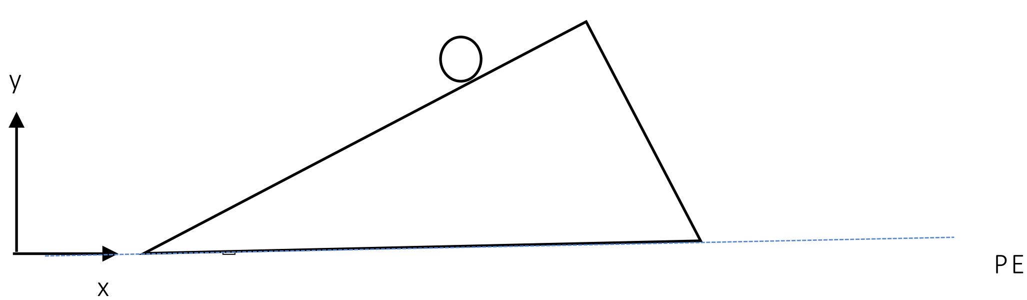
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \sum_k \lambda_k(t) \frac{\partial g_k(q_i)}{\partial \dot{q}_i} = 0$$

(01)

Now we have $N + K_c$ equations if we include the equations of constraint, for N coordinates and K_c unknown Lagrangian multipliers. These can be solved by first eliminating the terms containing the multipliers, then solving for the coordinates, and finally using those solutions to determine the forces of constraint, which are the expressions in the last term (the sum) of equation (01).

- **Example 01**

Consider a (point) particle of mass of m sliding on a wedge of angle ɵ. Find the total force of constraints using cartesian coordinates



Total kinetic energy of the particle; $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

Equation of constraints; $g_1(x, y) = y - \mathrm{x} \tan\theta$

This equation of constraint satisfies the condition of $g_i = 0$. So, the new Lagrangian is,

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda_1 g_1(x, y)$$

Considering equation 01 in x direction,

$$m\ddot{x} = \frac{\partial \mathcal{L}}{\partial x} = 0 + \lambda_1 \frac{\partial g_1(x, y)}{\partial x} = \lambda_1(-\tan\theta)$$

$$\Rightarrow \ddot{x} = -\frac{\lambda_1 \tan\theta}{m}$$

In y direction,

$$m\ddot{y} = \frac{\partial \mathcal{L}}{\partial y} = -mg + \lambda_1 \frac{\partial g_1(x, y)}{\partial y} = -mg + \lambda_1$$

Equation of constraint is $g_1(x, y) = y - \mathrm{x}\tan\theta = 0$. Which gives $y = \mathrm{x}\tan\theta$.

Then, $\ddot{y} = \ddot{\mathrm{x}} \tan\theta$

Combining constraint equation and equation for x direction;

$\frac{m\ddot{y}}{\tan\theta} = -\lambda_1 \tan\theta$. then,

$m\ddot{y} = -\lambda_1 \, tan^2\theta$

Plugging this into y equation, $-\lambda_1 \, tan^2\theta = -mg + \lambda_1$

$\lambda_1(1 + tan^2\theta) = mg \Rightarrow \lambda_1 = mg \; cos^2\theta$

Therefore,

$\ddot{y} = -g \, sin^2\,\theta$ and $\ddot{x} = -\mathrm{g} \sin\theta \cos\theta$

Force of constraint in y direction; $\boldsymbol{F}_y^c = \lambda_1 \frac{\partial g_1(x,y)}{\partial y} = \lambda_1 \times 1 = mg \, cos^2\,\theta$

Force of constraint in x direction:

$$\boldsymbol{F}_x^c = \lambda_1 \frac{\partial g_1(x,y)}{\partial x} = -\lambda_1 \times tan\,\theta = -mg \, cos^2\,\theta = -mg \, sin\theta \, cos\theta$$

Total force of constraint

$$|\boldsymbol{F}^c| = \sqrt{\left(\boldsymbol{F}_y^c\right)^2 + (\boldsymbol{F}_x^c)^2} = mg \, cos\theta$$

A more general case of a Lagrangian with a velocity-dependent potential energy is the particle of mass m with a charge q in an electromagnetic field. The Lagrangian of the system can be written as,

$$\mathcal{L} = \frac{m}{2}\dot{\vec{r}}^2 - \left(q\Phi(r) - q\,\vec{A}\cdot\dot{\vec{r}}\,\right)$$

Here, $\Phi(r)$ = electrostatic potential

$\vec{A}$ = vector potential

$\vec{B}$ = $\nabla \times \vec{A}$ also, $\vec{E} = -\nabla\,\Phi(r) - \frac{\partial \vec{A}}{\partial t}$

Now Euler Lagrange equation,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_i} = \frac{\partial \mathcal{L}}{\partial r_i}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_i} = m\ddot{r}_i + \frac{d}{dt}q\overrightarrow{A_\iota} = m\ddot{r}_i + q\frac{\partial \overrightarrow{A_\iota}}{\partial t} + q\sum_j \frac{\partial \overrightarrow{A_\iota}}{\partial r_j}\dot{r}_j$$

$\frac{\partial \mathcal{L}}{\partial r_i} = -q\left(\frac{\partial \Phi(r)}{\partial r_i} - \sum_j \frac{\partial \overrightarrow{A_J}}{\partial r_i}\,\dot{r}_j\right)$

$m\ddot{r}_i + q\frac{\partial \overrightarrow{A_\iota}}{\partial t} + q\sum_j \frac{\partial \overrightarrow{A_\iota}}{\partial r_j}\dot{r}_j = -q\left(\frac{\partial \Phi(r)}{\partial r_i} - \sum_j \frac{\partial \overrightarrow{A_J}}{\partial r_i}\,\dot{r}_j\right)$

Force component in i-direction,

$\vec{F}_i = -q\left(\frac{\partial \Phi(r)}{\partial r_i} + \frac{\partial \overrightarrow{A_\iota}}{\partial t}\right) - q\sum_j\left(\frac{\partial \overrightarrow{A_\iota}}{\partial r_j}\dot{r}_j + \frac{\partial \overrightarrow{A_J}}{\partial r_i}\,\dot{r}_j\right) = qE_i + q\left(\frac{\partial \overrightarrow{A_J}}{\partial r_i} - \frac{\partial \overrightarrow{A_\iota}}{\partial r_j}\right)\dot{r}_j$

$= qE_i + \left(\dot{\vec{r}} \times (\nabla \times \vec{A})\right) = \dot{\vec{r}} \times \vec{B}$

- **Example 02**

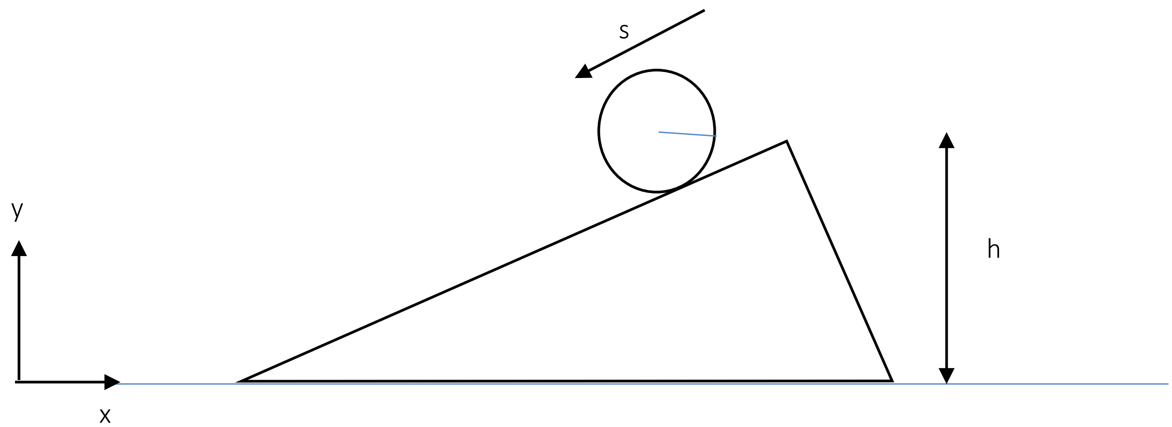
Consider a particle moving with a frictional force F = -ŋV.

New lagrangean is; $\mathcal{L}(q_i, \dot{q}_i, t) - \eta\dot{q}_i^{\,2}$

So the Euler lagrange equation of motion is; $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} - \eta\dot{q}_i$

- **Example 03**

Consider of ring with radius R, rolling along a wedge. Take s as the distance along the surface and Φ be the angle of rotation of the ring.



$\ddot{s} = \sqrt{\ddot{x}^2 + y^2}$

Equation of constraints; $g_1 = \left(\dot{s} - R\dot{\phi}\right)$. This is an example of a NON-holonomic constraint, since it is expressed in terms of velocities, not coordinates. (However, it would be easy to integrate this condition to make it holonomic – but we want to study the general mechanism here). This equation satisfies the condition $g_i = 0$.

Kinetic energy; $T = \frac{m}{2}\dot{s}^2 + \frac{I\dot{\phi}^2}{2}$ and potential energy; V = $(h - s\ sin\theta)mg$

Here I is the moment of inertia of the ring; I= mR^2

Lagrange equation is;

$$\mathcal{L} = \frac{m}{2}\dot{s}^2 + \frac{I\dot{\phi}^2}{2} - (h - s\ sin\theta)mg + \lambda_1 g_1$$

Euler Lagrange equation of motion for s- constraint is now,

$$m\ddot{s} = mg\ sin\theta + \lambda_1 \frac{\partial g_1}{\partial \dot{s}} = mg\ sin\theta + \lambda_1$$

Euler Lagrange equation of motion for Φ- constraint is now,

$$I\ddot{\phi} = -\lambda_1\ R$$

From constraint equation, $\ddot{\phi} = \frac{\ddot{s}}{R}$

Now we can write, $\frac{\ddot{s}}{R} = -\frac{\lambda_1 R}{I}$

$$m\left(-\frac{\lambda_1 R}{I}\right) = mg\ sin\theta + \lambda_1$$

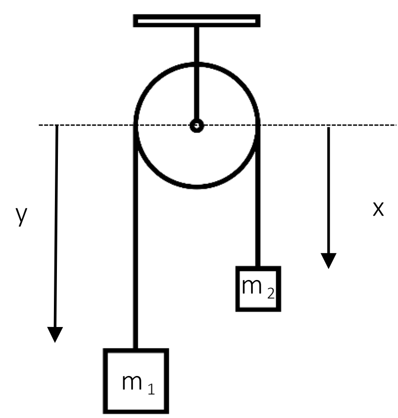
$$m\ddot{s} = mg\ sin\theta + \left(\frac{\ddot{s}I}{R^2}\right)$$

Finally, $\left(m + \frac{I}{R^2}\right)\ddot{s} = (2m)\ddot{s} = mg\ sin\theta$

This means that the acceleration of the ring down the ramp will be only ½ that of the point mass, since half of the work done by gravity goes into the kinetic energy of rotation of the ring and only ½ into the acceleration of the center of mass.

We can also interpret λ_1 as the frictional force along the edge of the ring that keeps it from slipping.

Homework: (Atwood's machine):



Equation of constraints; $g_1 = y + x - L = 0$ (holonomic constraints) where L is the total length of the string.

Therefore, we can write, $T = \frac{1}{2}(m_2\dot{x}^2 + m_1\dot{y}^2)$

Potential energy $U = -(m_1 y + m_2 x)g$

$$\mathcal{L} = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{x}^2 + (m_1 y + m_2 x)g + \lambda g_1$$

Using ELE in x direction, $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x}$

We get $m_2\ddot{x} = m_2 g + \lambda$

ELE in y direction, $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y}$

We get $m_1\ddot{y} = m_1 g + \lambda$

But, from constraint equation, $\dot{y} = -\dot{x}$

Which gives $\ddot{x} = \frac{(-m_1+m_2)g}{(m_1+m_2)}$ and $\ddot{y} = \frac{(m_1-m_2)g}{(m_1+m_2)}$

$$\lambda = -\frac{2m_1 m_2 g}{(m_1 + m_2)}$$

Forces of constraints $F_y = F_x = \lambda = -\frac{2m_1 m_2 g}{(m_1+m_2)}$

Which is equal to the tension in the string.