Week 11 Lecture 1 Notes

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1 Canonical Transformations and Type 1 Generating Functions

We can either express the $2N$ new generalized coordinates and conjugate momenta in terms of old coordinates (q_i, p_i) or new ones (Q_i, P_i). We assume the relationships can be inverted so that we can pick any $2N$ subset of old or new phase space coordinates and express the remaining $2N$ ones in terms of these.

$$\vec{\eta} = \begin{bmatrix} q_i \\ . \\ . \\ p_i \\ . \\ . \end{bmatrix} \rightarrow \vec{\zeta} = \begin{bmatrix} Q_i \\ . \\ . \\ P_i \\ . \\ . \end{bmatrix} \tag{1}$$

So we can switch between the Hamiltonian into the Kameltonian

$$H(\vec{\eta}) \rightarrow K(\vec{\zeta}) \tag{2}$$

$$\dot{\vec{\eta}} = J\vec{\nabla}_\eta H \rightarrow \dot{\vec{\zeta}} = J\vec{\nabla}_\zeta K \tag{3}$$

Harmonic Oscillator example:

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \tag{4}$$

where $q = x$

$$q = \frac{f(P)}{m\omega} sin(Q) \tag{5}$$

$$p = f(P) cos(Q) \tag{6}$$

Then

$$K = \frac{1}{2m} f(P)^2 \tag{7}$$

$P =$ constant since Q is (apparently) ignorable.

Now by definition

$$\dot{Q} = \frac{\partial K}{\partial P} \tag{8}$$

$$Q(t) = \frac{\partial K}{\partial P} t + \phi_0 \tag{9}$$

First Method:

$$\frac{df}{dt} + \sum P_i \dot{Q}_i - K(Q, P, t) = \sum p_i \dot{q}_i - H(q, p, t) \tag{10}$$

This works because both sides of the equation fulfill Hamilton Principle of Action.

We need to express

$$\vec{\eta}(\vec{\zeta}) \tag{11}$$

or

$$\vec{\zeta}(\vec{\eta}) \tag{12}$$

We need to make a function of one half of the old sets of variables and one half of the new sets of variables

Type 1 Generating Function

$$F_1(q, Q, t) \tag{13}$$

We can see this depends on old variable q and new variable Q.

Then

$$\frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F}{\partial t} + \sum P_i \dot{Q}_i - K(Q, P, t) = \sum p_i \dot{q}_i - H(q, p, t) \tag{14}$$

Examining the coefficients of Q_i

$$\dot{q}_i : \frac{\partial F_1}{\partial q_i} = p_i \tag{15}$$

$$\dot{Q}_i : \frac{\partial F_1}{\partial Q_i} + P_i = 0 \tag{16}$$

Therefore

$$P_i = -\frac{\partial F_1}{\partial Q_i} \tag{17}$$

When we look at everything else, it is easy to see that

$$\frac{\partial F}{\partial t} - K = -H \tag{18}$$

Therefore

$$K = H + \frac{\partial F}{\partial t} \tag{19}$$

If F does not depend on t the K and H are the same in value however they are still totally different functions because they depend on different variables.

Example:

$$F_1 = \sum_i q_i Q_i \tag{20}$$

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i \tag{21}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i \tag{22}$$

Ex:Ex:

$$H = \frac{p^2}{2m} + mgq \tag{23}$$

$$K = \frac{Q^2}{2m} - mgP \tag{24}$$

$$\frac{\partial K}{\partial Q} = \frac{Q}{m} = -\dot{P} \tag{25}$$

$$\frac{\partial K}{\partial P} = -mg = \dot{Q} \tag{26}$$

Which we know is the force.

Harmonic Oscillator

$$p = f(P)cos(Q) \tag{27}$$

We can take the ratio of q and p to get rid of $f(P)$

$$\frac{q}{p} = \frac{f(P)sin(Q)/m\omega}{f(P)cos(Q)} \tag{28}$$

Then

$$\frac{p}{q} = m\omega cot(Q) \tag{29}$$

$$p = qm\omega cot(Q) = \frac{\partial F_1}{\partial q} \tag{30}$$

Integrating to solve for F_1 we can see that

$$F_1 = \frac{q^2}{2} m\omega cot(Q) + G(Q) \tag{31}$$

Then by definition

$$P = -\frac{\partial F_1}{\partial Q} = \frac{q^2 m\omega}{2} \frac{1}{sin^2(Q)} + G'(Q) \tag{32}$$

From Eq. (5):

$$f(P) = \frac{m\omega q}{sin(Q)} \tag{33}$$

We can make a guess $f(P) = \sqrt{2m\omega P}$, $G = 0$
In general :

$$P(q, p) \tag{34}$$

and

$$Q(q, p) \tag{35}$$

are canonical only if

$$\frac{\partial^2 F_1}{\partial Q \partial q} = \frac{\partial p}{\partial q} \tag{36}$$

Which must be equal to

$$\frac{\partial^2 F_1}{\partial q \partial Q} = -\frac{\partial P}{\partial q} \tag{37}$$

Plugging into Eq. (7):

$$K = \omega P \tag{38}$$

$P = constant$, $K = constant$ and therefore by the above equation $\omega = constant$.
Now we may be wondering, What is this ω thing?

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \tag{39}$$

$$q = \sqrt{\frac{2P}{m\omega}} sin(\omega t + \phi_0) \tag{40}$$

$$p = \sqrt{2m\omega P} cos(\omega t + \phi_0) \tag{41}$$

From our previous experience, we know that this is indeed the behavior of a harmonic oscillator with (position) amplitude A. Therefore,

$$\sqrt{\frac{2P}{m\omega}} = A \tag{42}$$

So

$$P = \frac{m\omega A^2}{2} \tag{43}$$

$$K = \frac{m\omega^2 A^2}{2} \tag{44}$$

which is indeed the energy of a harmonic oscillator.

We may be wondering how we know that a transformation from $\vec{\eta}$ to $\vec{\zeta}$ is actually canonical. There is a direct way to check whether the transformation is canonical using the symplectic formulation.

$$\vec{\zeta}(\vec{\eta}) \to (\dot{\vec{\zeta}})_i = \sum_{j=1}^{2N} \frac{\partial \zeta_i}{\partial \eta_j} \dot{\eta}_j \tag{45}$$

We define

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j} \tag{46}$$

So

$$\begin{aligned} (\dot{\vec{\zeta}})_i = M_{ij} \dot{\vec{\eta}_j} = M_{ij} J_{jk} (\vec{\nabla}_\eta H(\vec{\eta}))_k = M_{ij} J_{jk} \frac{\partial \zeta_l}{\partial \eta_k} (\vec{\nabla}_\zeta K(\vec{\zeta}))_l \\ = M_{ij} J_{jk} M_{lk} (\vec{\nabla}_\zeta K(\vec{\zeta}))_l = M_{ij} J_{jk} M^T_{kl} (\vec{\nabla}_\zeta K(\vec{\zeta}))_l \end{aligned} \tag{47}$$

We know the value of H and K is the same but K is H with all variables replaced.

$$\dot{\vec{\zeta}} = J \vec{\nabla}_\zeta K \to J = M^T J M \tag{48}$$

This last equation (or its equivalent with M and M^T exchanged) is the symplectic condition.