Central Force Problem: Kepler. PHYS 603, Classical Mechanics, Dr. Kuhn. Lecture Notes from 7 February, 2019

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Recall the energy equation from the last lecture:

$$h = \frac{\mu}{2}\dot{r}^2 + \frac{P_\phi^2}{2\mu r^2} + V(r) = E \qquad (1)$$

Also recall that we designated

$$\text{"}V(r)\text{"} = \frac{P_\phi^2}{2\mu r^2} + V(r)$$

Now we'll bring in the concept of gravitational attraction:

$$V(r) = -G\frac{M\mu}{r} = -\frac{k}{r}$$

For brevity's sake let $k = GM\mu$.

We'll assume all motion is in the x-y plane. Rearranging $\dot{r}$:

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = r'\frac{P_\phi}{\mu r^2} = \frac{P_\phi}{\mu}(-\frac{\partial}{\partial\phi}\frac{1}{r})$$

Now make the variable transformation

$$u = \frac{1}{r} \Rightarrow \dot{r} = \frac{P_\phi}{\mu}u'$$

Now rewriting (1):

$$\frac{P_\phi^2}{2\mu}u'^2 + \frac{P_\phi^2}{2\mu}u^2 - ku = E$$

Rearranging:

$$u'^2 = \frac{2\mu}{P_\phi^2}(E + ku) - u^2$$

Now substituting this expression into the integral from the previous lecture:

$$\int_{u(\phi_0)}^{u(\phi)} \frac{du}{\sqrt{\frac{2\mu E}{P_\phi^2} + \frac{2\mu k}{P_\phi^2}u - u^2}} = \int_{\phi_0}^{\phi} d\phi'$$

Note that $E \to \infty$ as $r \to 0$. Also note we are assuming $P_\phi \neq 0$. The maximum value of u occurs where $u' = 0$. Designate this value u_m. With $u' = 0$ our energy expression becomes:

$$\frac{2\mu}{P_\phi^2}(E + ku_m) - u^2 = 0$$

Completing the polynomial square:

$$\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4} - (u_m - \frac{\mu k}{P_\phi^2})^2 = 0$$

Solving for u_m:

$$u_m = \frac{\mu k}{P_\phi^2} \pm \sqrt{\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4}}$$

Where the max occurs with the + sign active in the expression. We'll set $u(\phi_0) = u_m$ and $\phi_0 = 0$, so $u(0) = u_m$.

Now returning to the integral. Define $v = u - \frac{\mu k}{P_\phi^2}$. Rewriting the integral:

$$\int_{u_m - \frac{\mu k}{P_\phi^2}}^{u(\phi) - \frac{\mu k}{P_\phi^2}} \frac{dv}{\sqrt{\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4} - v^2}} = \int_{\phi_0}^{\phi} d\phi'$$

Now define $w = \frac{v}{\sqrt{\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4}}}$. And again rewriting the integral:

$$\int_{1}^{w(\phi)} \frac{dw}{\sqrt{1 - w^2}} = \int_{\phi_0}^{\phi} d\phi'$$

Reversing the limits of integration:

$$-\int_{w(\phi)}^{1} \frac{dw}{\sqrt{1 - w^2}} = \int_{\phi_0}^{\phi} d\phi'$$

$$\arccos(1) - \arccos(w(\phi)) = \phi$$

$$\cos\phi = \frac{u(\phi) - \frac{\mu k}{P_\phi^2}}{\sqrt{\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4}}}$$

$$u(\phi) = \frac{\mu k}{P_\phi^2}(1 + \cos\phi\sqrt{1 + \frac{2EP_\phi^2}{\mu k^2}}\ )$$

Now define $c = \frac{\mu k}{P_\phi^2}$ and $e = \sqrt{1 + \frac{2EP_\phi^2}{\mu k^2}}$, making the expression now:

$$u(\phi) = c(1 + e\, cos\phi)$$

Recall $u = \frac{1}{r}$:

$$\boxed{r(\phi) = \frac{1}{c(1 + e\, cos\phi)}}$$

Now we'll explore the impact manipulating the variable e has on the shape of the function $r(\phi)$. Note they are all conic sections.

Case 1
$e = 0$
Circle

$$r(\phi) = \frac{1}{c}$$

$$c = \frac{\mu k}{P_\phi^2}$$

recall $k = GM\mu$:

$$r(\phi) = \frac{1}{c} = \frac{P_\phi^2}{\mu k} = \frac{P_\phi^2}{GM\mu^2}$$

Note $r(\phi)$ is a constant, and since $\dot{\phi} = \frac{P_\phi}{\mu r^2} \Rightarrow \dot{\phi}$ is constant.

And since $e = \sqrt{1 + \frac{2EP_\phi^2}{\mu k^2}}$, $e = 0 \Rightarrow E = -\frac{\mu k}{2P_\phi^2}$, the minimum energy the system can have without a complex solution.

Case 2
$e = 1$
Parabola

$$r(\phi) = \frac{1}{c(1 + e\, cos\phi)} = \frac{1}{c(1 + cos\phi)}$$

$$e = \sqrt{1 + \frac{2EP_\phi^2}{\mu k^2}}; \quad e = 1 \Rightarrow E = 0$$

Case 3
$e > 1$
Hyperbola

$$r(\phi) = \frac{1}{c(1 + e\,cos\phi)}$$

$$e = \sqrt{1 + \frac{2EP_\phi^2}{\mu k^2}}; \quad e > 1 \Rightarrow E > 0$$

More to follow on the hyperbola in the next lecture.

Case 4
$0 < e < 1$
Ellipse

$$r(\phi) = \frac{1}{c(1 + e\,cos\phi)}$$

$$r_{min} = \frac{1}{c(1 + e\,cos\phi)}; \quad r_{max} = \frac{1}{c(1 - e\,cos\phi)}$$

Solving for the semi-major axis, a:

$$2a = \frac{1}{c(1-e)} + \frac{1}{c(1+e)}$$

$$a = \frac{1}{c(1-e^2)}$$

Semi-minor axis, b:

$$b = a\sqrt{1-e^2}$$