PHYSICS 603 - Classical Mechanics

Hamiltonian Mechanics

Md Aziz Ar Rahman

April 3, 2020

Let consider $\mathcal{L}$ be the Lagrangian of a system where $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t)$ with $i = 1, 2, \ldots, k$.
We know that,

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

We defined a h function such that,

$$h = \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t) \tag{0.1}$$

$$h = h(q_i, \dot{q}_i, t) = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t) \tag{0.2}$$

h function may or may not represent energy.
Now,

$$\begin{aligned} dh &= \sum_i (p_i d\dot{q}_i + \dot{q}_i dp_i) - \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i\right) - \frac{\partial \mathcal{L}}{\partial t} dt \\ &= \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \sum_i p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt \\ &\qquad \left[\text{Where, } \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \dot{p}_i\right] \\ \therefore dh &= \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \frac{\partial \mathcal{L}}{\partial t} dt \end{aligned}$$

It is clear that variation of h depends on variation in p_i, q_i and possibly of t.

The Hamiltonian H and the energy function h has the same value and H can be constructed in the same manner of equation 0.2.

So, $H(q_i, p_i, t) = h[q_i, \dot{q}_i(q_j, p_j, t), t]$
And we can obtain the following equations,

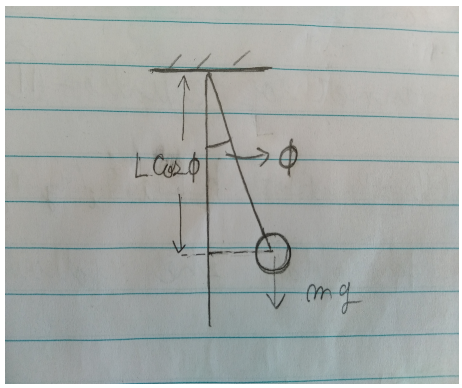
$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \frac{\partial h}{\partial p_i} = \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= \frac{\partial h}{\partial q_i} = -\dot{p}_i \\ \frac{\partial H}{\partial t} &= \frac{\partial h}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \end{aligned}$$

Now we can write,

$$\frac{\partial H}{\partial q_i} = -\dot{p_i} \qquad \text{and} \qquad \frac{\partial H}{\partial p_i} = \dot{q_i}$$

Which are known as Canonical equations of Hamilton.

Example: Let's consider a simple pendulum of mass m and length L.



The Lagrangian,

$$\mathcal{L}(\phi, \dot{\phi}) = \frac{1}{2} m L^2 \dot{\phi}^2 - mgL(1 - \cos\phi)$$

$$\text{Now, } p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mL^2 \dot{\phi} \tag{0.3}$$

So the h function can be written as,

$$\begin{aligned} h &= p_\phi \dot{\phi} - \mathcal{L} \\ &= mL^2\dot{\phi}^2 - \frac{1}{2} mL^2 \dot{\phi}^2 + mgL(1 - \cos\phi) \\ &= \frac{1}{2} mL^2 \dot{\phi}^2 + mgL(1 - \cos\phi) \\ &= \frac{{p_\phi}^2}{2mL^2} + mgL(1 - \cos\phi) \\ &= T + V \\ &= H(\phi, p_\phi) \end{aligned}$$

Now the Canonical equations of Hamilton becomes,

$$\dot{p_\phi} = -\frac{\partial H}{\partial \phi} = -mgL\sin\phi \tag{0.4}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mL^2} \tag{0.5}$$

Using the definition of equation 0.3, 0.4 becomes,

$$
\begin{aligned}
mL^2\ddot{\phi} =& - mgL\phi \\
& \text{[Using small angle approximation]} \\
\therefore \ddot{\phi} =& - \frac{g}{L}\phi
\end{aligned}
$$

Which gives the equation of motion of simple pendulum with angular frequency $\omega = \sqrt{\frac{g}{L}}$.

Let's think about an object going upward with speed $\dot{y}$.

$$
\begin{aligned}
\text{So Lagrangian, } \mathcal{L}(y, \dot{y}) =& \frac{1}{2}m\dot{y}^2 - mgy \\
\therefore p_y =& \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}
\end{aligned}
$$

So the h function is,

$$
\begin{aligned}
h =& p_y\dot{y} - \mathcal{L}(y, \dot{y}) \\
=& m\dot{y}^2 - \frac{1}{2}m\dot{y}^2 + mgy \\
=& \frac{1}{2}m\dot{y}^2 + mgy \\
=& \frac{{p_y}^2}{2m} + mgy = E = H(y, p_y, t)
\end{aligned}
$$

$$
\begin{aligned}
&\text{So, } \frac{\partial H}{\partial p_y} = \dot{y} \implies \frac{p_y}{m} = \dot{y} \\
&\text{And,} \\
&\frac{\partial H}{\partial y} = - \dot{p_y} \\
&\implies \dot{p_y} = - mg \\
&\therefore p_y = - mgt + p_{y0} \\
&\implies m\dot{y} = - mgt + p_{y0} \\
&\therefore \dot{y} = - gt + \frac{p_{y0}}{m} \\
&\therefore y(t) = y_0 - \frac{1}{2}gt^2 + \frac{p_{y0}}{m}t
\end{aligned}
$$

In general case we can write Lagrangian as,

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\dot{\bar{q}})^T\Pi(\dot{\bar{q}}) + (\dot{\bar{q}})^T(\bar{a}) + \mathcal{L}_0(\bar{q},t)\\ =& \frac{1}{2}\sum_{ij}\dot{q}_i\dot{q}_j\Pi_{ij} + \sum_i \dot{q}_i a_i + \mathcal{L}_0(q_i,t)\end{aligned}$$

$$\text{Where, } \dot{\bar{q}} = \begin{pmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \vdots\end{pmatrix}$$

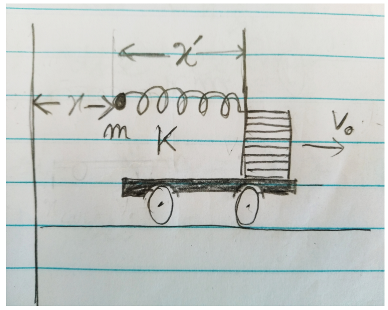
Π_{ij} and a_i are function of coordinate and possibly time.

$$\begin{aligned}\text{Now, } p_l =& \frac{\partial \mathcal{L}}{\partial \dot{q}_l} = \frac{1}{2}\sum_{ij}\dot{q}_i\delta_{jl}\Pi_{ij} + \frac{1}{2}\sum_{ij}\delta_{il}\dot{q}_j\Pi_{ij} + \sum_i \delta_{il}a_i\\ \therefore p_l =& \frac{1}{2}\sum_i \dot{q}_i\Pi_{il} + \frac{1}{2}\sum_j \dot{q}_j\Pi_{lj} + a_l\\ =& \frac{1}{2}\sum_i \dot{q}_i\Pi_{il} + \frac{1}{2}\sum_j \dot{q}_j\Pi_{jl} + a_l \qquad \because \Pi = \Pi^T\\ =& \sum_i \dot{q}_i\Pi_{il} + a_l\\ \therefore (\bar{p})^T =& ((\dot{\bar{q}})^T\Pi) + (\bar{a})^T \qquad (0.6)\\ \implies (\dot{\bar{q}})^T =& (\bar{p}-\bar{a})^T\Pi^{-1}\end{aligned}$$

Now,

$$\begin{aligned}H =& \sum_i p_i\dot{q}_i - \mathcal{L}\\ =& (\bar{p})^T(\dot{\bar{q}}) - \mathcal{L}\\ =& (\dot{\bar{q}})^T\Pi\dot{\bar{q}} + (\bar{a})^T\dot{\bar{q}} - \frac{1}{2}(\dot{\bar{q}})^T\Pi(\dot{\bar{q}}) - (\dot{\bar{q}})^T(\bar{a}) - \mathcal{L}_0(\bar{q},t)\\ =& \frac{1}{2}(\dot{\bar{q}})^T\Pi\dot{\bar{q}} - \mathcal{L}_0(\bar{q},t) \qquad \because (\bar{a})^T\dot{\bar{q}} = (\dot{\bar{q}})^T(\bar{a})\\ =& \frac{1}{2}(\bar{p}-\bar{a})^T\Pi^{-1}\Pi(\Pi^{-1})^T(\bar{p}-\bar{a}) - \mathcal{L}_0(\bar{q},t)\\ =& \frac{1}{2}(\bar{p}-\bar{a})^T(\Pi^{-1})(\bar{p}-\bar{a}) - \mathcal{L}_0(q,t) \qquad (\Pi^{-1})^T = \Pi^{-1}\\ =& H(\bar{q},\bar{p},t)\\ \therefore H(\bar{q},\bar{p},t) =& \frac{1}{2}(\bar{p}-\bar{a})^T(\Pi^{-1})(\bar{p}-\bar{a}) - \mathcal{L}_0(q,t) \qquad (0.7)\end{aligned}$$

Example: Let consider a mass is attached to a spring k in one end. The other end of the spring is attached to a mass less cart which is moving uniformly with speed v_0.



The Lagrangian,

$$\mathcal{L}(x,\dot{x},t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x-v_0t)^2$$

Comparing with general form of Lagrangian,

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\dot{\bar{q}}^T\Pi\dot{\bar{q}} + \dot{\bar{q}}^T\bar{a} + \mathcal{L}_0(q,t)\\ =& \frac{1}{2}\sum_{ij}\dot{q}_i\dot{q}_j\Pi_{ij} + \sum_i \dot{q}_i a_i + \mathcal{L}_0(q,t)\end{aligned}$$

This is one dimensional case,
So,

$$\begin{aligned}(\dot{\bar{q}}) =& (\dot{x}), \Pi = m \qquad \text{and} \qquad \bar{a} = 0,\ \mathcal{L}_0 = -\frac{1}{2}k(x-v_0t)^2\\ \text{Now, } p_x =& \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}\\ \therefore (\bar{p}) =& p_x = m\dot{x}\end{aligned}$$

So according to equation 0.7,

$$\begin{aligned}H(x,p_x,t) =& \frac{1}{2}p_x\frac{1}{m}p_x + \frac{1}{2}k(x-v_0t)^2\\ =& \frac{{p_x}^2}{2m} + \frac{1}{2}k(x-v_0t)^2\end{aligned}$$

So canonical equations of Hamilton,

$$\begin{aligned}
\frac{\partial H}{\partial x} =& k(x - v_0 t) = -\dot{p}_x \\
\frac{\partial H}{\partial p_x} =& \frac{p_x}{m} = \dot{x} \\
\therefore \dot{p}_x =& -k(x - v_0 t) \\
\text{and } \dot{x} =& \frac{p_x}{m}
\end{aligned}$$

are the equation of motion. Again H is explicitly dependent on t, so energy is not conserved.

Lets see the figure again.
If we define position of the mass by x' then the Lagrangian can be written as,

$$\begin{aligned}
\mathcal{L} =& T - V \\
=& \frac{1}{2}m(\dot{x}' + v_0)^2 - \frac{1}{2}kx'^2 \\
\text{Where, } T =& \frac{1}{2}m(\dot{x}' + v_0)^2 \\
V =& \frac{1}{2}kx'^2 \\
\text{So, } \mathcal{L} =& \frac{1}{2}m\dot{x}'^2 + m\dot{x}'v_0 + \frac{1}{2}mv_0^2 - \frac{1}{2}kx'^2 \\
\therefore \frac{\partial \mathcal{L}}{\partial \dot{x}'} =& p = m\dot{x}' + mv_0 \\
\text{Here, } \mathcal{L}_0 =& \frac{1}{2}mv_0^2 - \frac{1}{2}kx'^2
\end{aligned}$$

So, following the equation 0.7

$$\begin{aligned}
H(x', p) =& \frac{1}{2}(p - mv_0)\frac{1}{m}(p - mv_0) - \frac{1}{2}mv_0^2 + \frac{1}{2}kx'^2 \\
=& \frac{1}{2}\frac{(p - mv_0)^2}{m} + \frac{1}{2}kx'^2 - \frac{1}{2}mv_0^2
\end{aligned}$$

Here, H is independent of time. H doesn't represent total energy E, but it is a conserved quantity. Except for the last constant term $\frac{1}{2}mv_0^2$, H represents total energy of the mass due to its motion relative to the moving cart.

$$\begin{aligned}
\text{Now, } \frac{\partial H}{\partial x'} =& kx' = -\dot{p} \\
\frac{\partial H}{\partial p} =& \frac{(p - mv_0)}{m} = \dot{x}' \\
\therefore \dot{p} =& -kx' \quad \text{and} \quad \frac{p}{m} - v_0 = \dot{x}' \\
\implies \ddot{x}' =& \frac{\dot{p}}{m} = -\frac{k}{m}x'
\end{aligned}$$

which represents oscillatory motion with frequency $\sqrt{\frac{k}{m}}$.

Lets consider a charged particle of mass m and charge e moving through an EM field. The Lagrangian of that particle is,

$$\mathcal{L} = \frac{1}{2}m\dot{\bar{r}}^2 - e\phi(\bar{r}, t) + e\dot{\bar{r}}.\bar{A}(\bar{r}, t)$$

Where $\phi(\bar{r}, t)$ is electric potential and $\bar{A}(\bar{r}, t)$ is vector potential.

$$\text{Now, } (\bar{r}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, (\dot{\bar{r}}) = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$(\bar{A}) = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Now the Lagrangian can be written as,

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + e(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - e\phi(\bar{r}, t) \\
=& \frac{1}{2}\begin{pmatrix} \dot{x} & \dot{y} & \dot{z} \end{pmatrix} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} + \begin{pmatrix} \dot{x} & \dot{y} & \dot{z} \end{pmatrix} \begin{pmatrix} eA_x \\ eA_y \\ eA_z \end{pmatrix} - e\phi(\bar{r}, t) \\
\therefore \mathcal{L}(\bar{r}, \dot{\bar{r}}, t) =& \frac{1}{2}(\dot{\bar{r}})^T \Pi \dot{\bar{r}} + (\dot{\bar{r}})^T(\bar{a}) + \mathcal{L}_0(\bar{r}, t) \qquad (0.8) \\
\text{Where, } (\bar{a}) =& e(\bar{A}) \qquad \Pi = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}
\end{aligned}$$

$$\mathcal{L}_0(\bar{r}, t) = -e\phi(\bar{r}, t)$$

According to 0.6;

$$(\bar{p}) = \Pi(\dot{\bar{r}}) + (\bar{a}) = \Pi(\dot{\bar{r}}) + e\bar{A} \qquad \because \Pi^T = \Pi$$

In other words, $p_i = m\dot{r}_i + eA_i$

where, $i = x, y, z$

$\therefore \bar{p} = m\dot{\bar{r}} + e\bar{A}$ which is known as Canonical Momentum .

Now equation 0.7 becomes,

$$\begin{aligned}
H(\bar{r}, \bar{p}, t) =& \frac{1}{2}(\bar{p} - e\bar{A})^T \Pi^{-1} (\bar{p} - e\bar{A}) - \mathcal{L}_0(\bar{r}, t) \\
=& \frac{1}{2}(\bar{p} - e\bar{A})^T \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}^{-1} (\bar{p} - e\bar{A}) + e\phi(\bar{r}, t) \\
=& \frac{1}{2}(\bar{p} - e\bar{A})^T \frac{1}{m} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\bar{p} - e\bar{A}) + e\phi(\bar{r}, t) \\
=& \frac{1}{2m} \begin{pmatrix} p_x - eA_x & p_y - eA_y & p_z - eA_z \end{pmatrix} \begin{pmatrix} p_x - eA_x \\ p_y - eA_y \\ p_z - eA_z \end{pmatrix} + e\phi(\bar{r}, t) \\
=& \frac{1}{2m}[(p_x - eA_x)^2 + (p_y - eA_y)^2 + (p_z - eA_z)^2] + e\phi(\bar{r}, t) \\
=& \frac{1}{2m}(\bar{p} - e\bar{A})^2 + e\phi = T + V
\end{aligned}$$

Here H represents Total Energy but the quantity is conserved if and only if ϕ an $\bar{A}$ are time independent.

Now the Canonical equations of Hamilton are as below,

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{1}{2m}[-2(p_x - eA_x)e\frac{\partial A_x}{\partial x} - 2(p_y - eA_y)e\frac{\partial A_y}{\partial x} - 2(p_z - eA_z)e\frac{\partial A_z}{\partial x}] - e\frac{\partial \phi}{\partial x}$$

$$\therefore \dot{p}_x = \frac{e}{m}[(p_x - eA_x)\frac{\partial A_x}{\partial x} + (p_y - eA_y)\frac{\partial A_y}{\partial x} + (p_z - eA_z)\frac{\partial A_z}{\partial x}] - e\frac{\partial \phi}{\partial x}$$

Similarly,

$$\begin{aligned}
\dot{p}_y =& \frac{e}{m}[(p_x - eA_x)\frac{\partial A_x}{\partial y} + (p_y - eA_y)\frac{\partial A_y}{\partial y} + (p_z - eA_z)\frac{\partial A_z}{\partial y}] - e\frac{\partial \phi}{\partial y} \\
\dot{p}_z =& \frac{e}{m}[(p_x - eA_x)\frac{\partial A_x}{\partial z} + (p_y - eA_y)\frac{\partial A_y}{\partial z} + (p_z - eA_z)\frac{\partial A_z}{\partial z}] - e\frac{\partial \phi}{\partial z}
\end{aligned}$$

And,

$$\begin{aligned}
\dot{x} =& \frac{\partial H}{\partial p_x} = \frac{p_x - eA_x}{m} \implies p_x = m\dot{x} + eA_x \\
\dot{y} =& \frac{\partial H}{\partial p_y} = \frac{p_y - eA_y}{m} \implies p_y = m\dot{y} + eA_y \\
\dot{z} =& \frac{\partial H}{\partial p_z} = \frac{p_z - eA_z}{m} \implies p_z = m\dot{z} + eA_z
\end{aligned}$$

Using the definition of $\dot{x}, \dot{y}$ and $\dot{z}$ we can write,

$$\begin{aligned}
\dot{p}_x =& e[\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}] - e\frac{\partial \phi}{\partial x} \\
\implies& m\ddot{x} + e\frac{dA_x}{dt} = e[\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}] - e\frac{\partial \phi}{\partial x} \\
\implies& m\ddot{x} + e\dot{x}\frac{\partial A_x}{\partial x} + e\dot{y}\frac{\partial A_x}{\partial y} + e\dot{z}\frac{\partial A_x}{\partial z} + e\frac{\partial A_x}{\partial t} = e\dot{x}\frac{\partial A_x}{\partial x} + e\dot{y}\frac{\partial A_y}{\partial x} + e\dot{z}\frac{\partial A_z}{\partial x} - e\frac{\partial \phi}{\partial x} \\
\implies& m\ddot{x} = e\dot{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) + e\dot{z}(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}) + e(-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}) \\
\implies& m\ddot{x} = e[\dot{y}(\bar{\nabla} \times \bar{A})_z - \dot{z}(\bar{\nabla} \times \bar{A})_y] + e[-\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}]_x \\
\implies& m\ddot{x} = e(\bar{E})_x + e(\dot{\bar{r}} \times \bar{B})_x \\
\therefore& (m\ddot{\bar{r}})_x = e[\bar{E} + (\dot{\bar{r}} \times \bar{B})]_x
\end{aligned}$$

$$\therefore m\ddot{\bar{r}} = e[\bar{E} + (\dot{\bar{r}} \times \bar{B})] \tag{0.9}$$

This is the equation of motion for the charged particle moving in EM field. The equation 0.9 also represents the force acting on the charged particle while moving through an EM field.