Phase space

- It is a 2N dimensional space of q_i (generalized coordinate) and p_i (generalized momenta).

In phase space we use $\eta_i(t)$,

$$\text{Where } \eta_{1-N} = q_{1-N}$$

$$\eta_{N-2N} = p_{1-N}$$

N+1- 2N

- Advantage of this formulation is that, since Hamiltonian equation of motion are 1st order differential equations if we know $\eta(t)$ at any time 't' then we know value of $\eta(t+dt)$, since:

$$\eta(t+dt) = \eta(t) + \dot{\eta}(t)\, .dt$$

$$\dot{\eta}(t) = J.\nabla\eta\, H(\eta, t) \text{ , where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Which gives Hamiltonian equations of motion.

- Once we know a point in phase space (value of p_i and $q_{i)}$ then we can know about present past and future of that point.

- Disadvantage: If N >1 then it becomes hard to draw phase space diagram. So, we will consider simple case only(N=1):

1. Free Motion:

   We have, Hamiltonian for free particle,

   $$H=\frac{p2}{2m}$$

   Then, Hamiltonian equation of motions are,

$$\dot{x} = p/m$$

$$dx/dt = p/m$$

$$dx = (p/m).dt$$

Similarly,

$$\dot{p} = 0$$

$$dp = 0$$

In phase the portrait will be

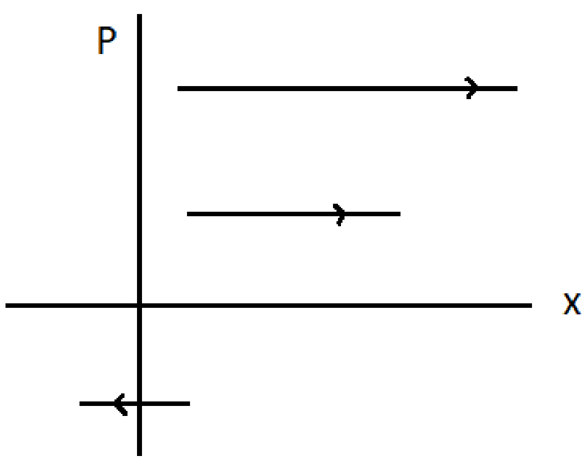


Fig: phase space diagram of free particle

2. Gravity:

The Hamiltonian for the particle in Gravitational Field is given by,

$$H = \frac{p2}{2m} + mgx$$

Then Hamiltonian equations of motion will reduce to,

$$dx = \frac{p}{m}\,dt \qquad \text{and} \qquad dp = -mg\,dt$$

The phase space diagram will be,

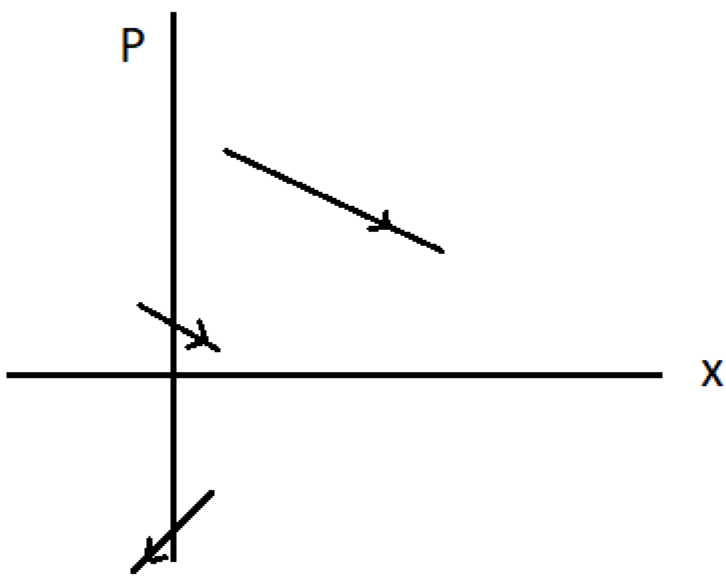


Fig: phase space diagram of particle in gravitational field

3. Harmonic oscillator:

   The Hamiltonian for harmonic oscillator is given by,

$$H = \frac{p^2}{2m} + k\frac{x^2}{2}$$

   The Hamiltonian equation of motion reduces to

$$dx = \frac{p}{m}dt \qquad \text{and} \qquad dp = -\,kx$$

   The phase space diagram will be then,

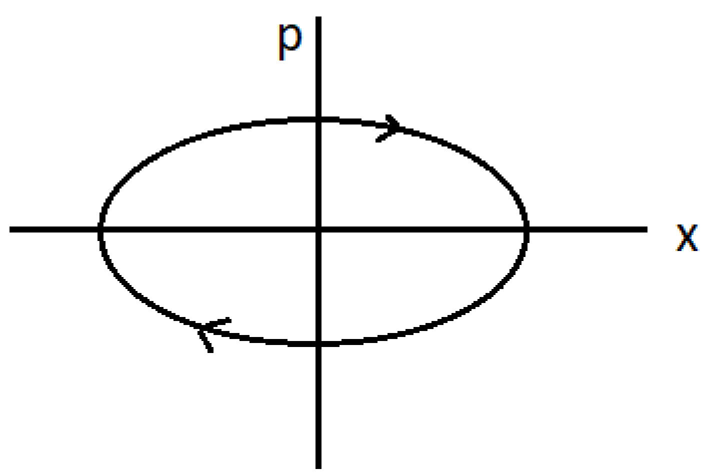


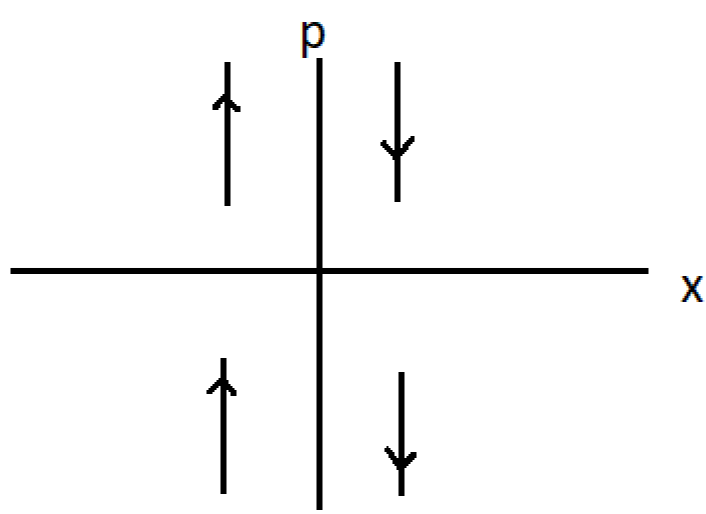
Fig: phase space diagram of harmonic oscillator

4. Lens:

   Since, interaction time is short , dx = 0 then

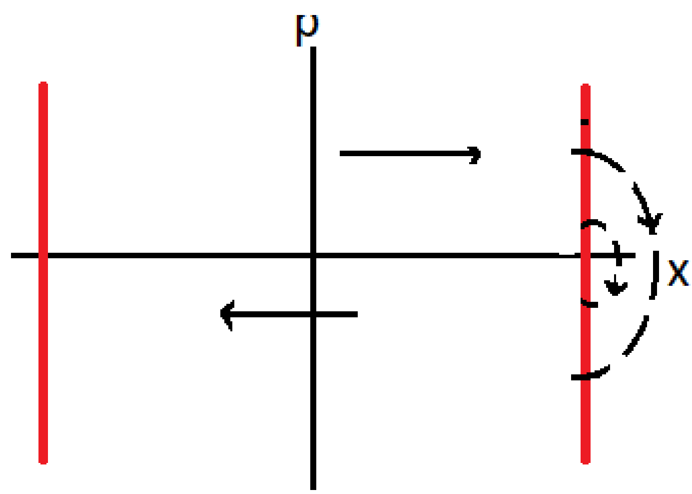
   $$dp = -kx$$

   The phase space diagram will be,



5. Particle in a box:

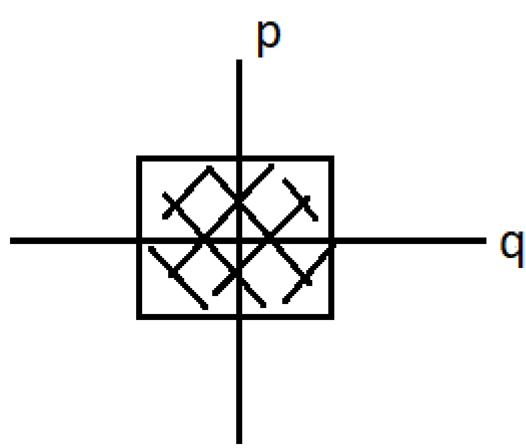
For particle in a box the phase space diagram will be,



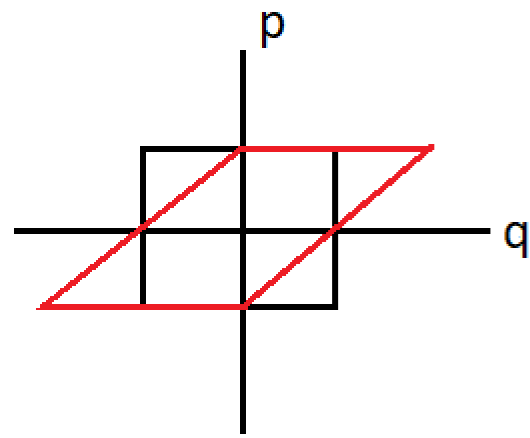
Red lines represents walls of container.

Due to elastic collision of particles on the wall of container the direction of momentum on wall of container after collision is interchanged which is represented by dotted line.

- Ensemble of particles can be represented by volume in phase space. It consists of continuous homogeneous distribution of ~~a point~~ points over a finite volume. The probability of finding particle inside the volume is constant and outside the volume is zero. Anywhere arbitrary close to the perimeter of the volume, there is particle inside. We consider the above discussed cases as ensemble of particles.

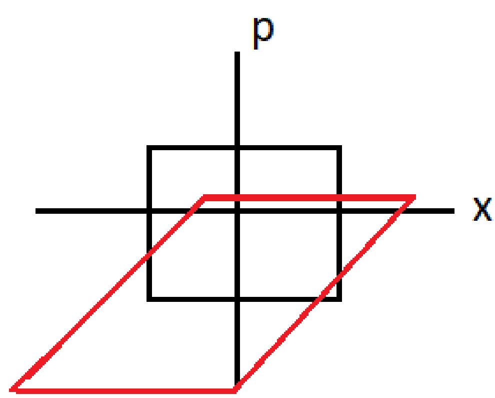


- Free motion:



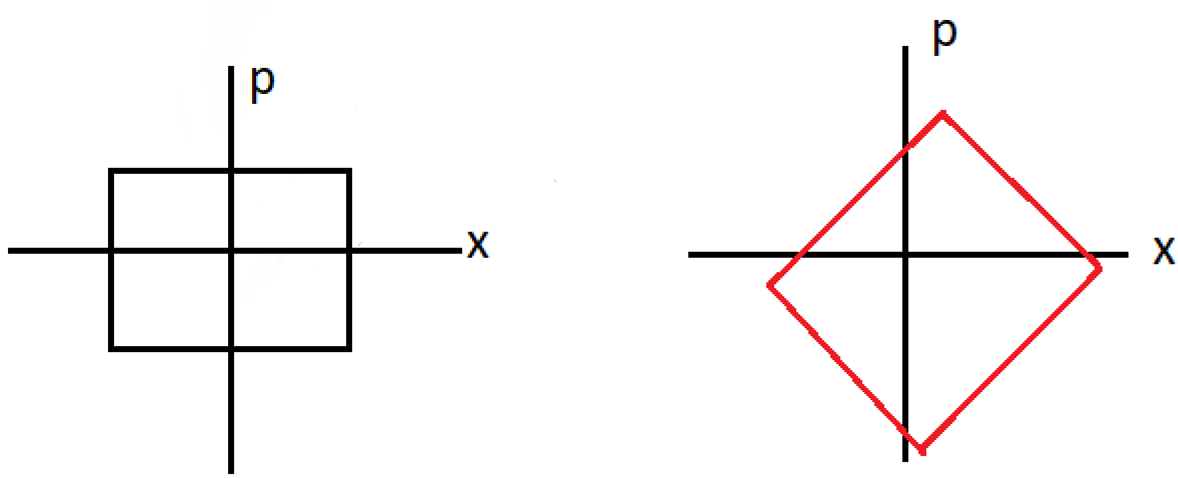
The black portion represents initial phase space portrait of ensemble of particles and as time passes the shape of ensemble changes to red portion but volume remains same.

- Gravity:



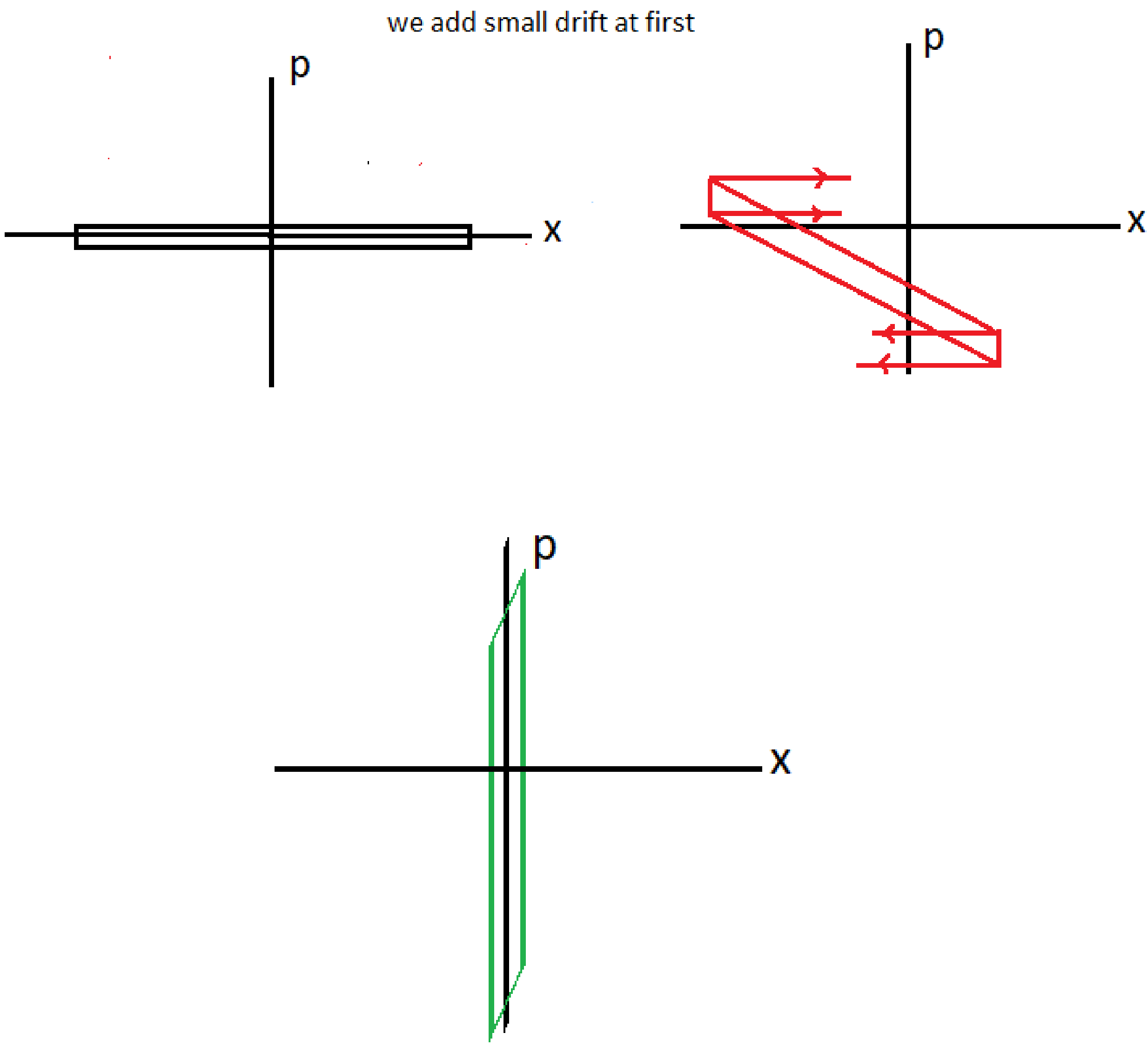
The black portion represents initial phase space portrait of ensemble of particles and as time passes the shape of ensemble changes to red portion but volume remains same.

- Harmonic oscillator:



The black portion represents initial phase space portrait of ensemble of particles and as time passes the shape of ensemble changes to red portion but volume remains same.

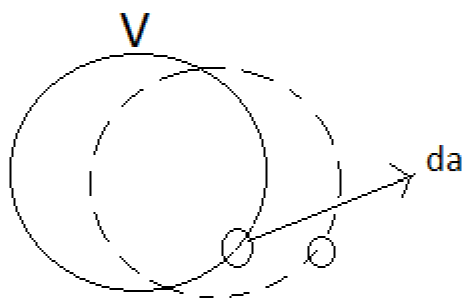
- Lens:



Liouville's Theorem:

It states that the density of system points in the vicinity of a given system point traveling through phase-space is constant with time.

It is assumed that Hamiltonian of all particles in the ensemble is same and they follow same equation of motion.



occupied phase space

As the time passes the ~~phase~~ evolves with time and change in volume ,

$$dV = \iint \dot{\eta} . da . dt$$

$$= \iiint \nabla\eta . \dot{\eta} . dv . dt \qquad \text{--------------(1)}$$

This is the divergence in eta-space of eta-dot

Since,

$\partial H/\partial q_i = -\partial p_i/\partial t$ --------------------(2)

We get,

$$\nabla\eta . \dot{\eta} = \sum(\partial/\partial q_i \,.\, \partial H/\partial p_i + \partial/\partial p_i \,.\, (-\partial H/\partial q_i))$$

$$= 0 \qquad \text{---------------(3)}$$

Hence from above equations we get after substituting in equation (1)

dv = 0

i.e, V= constant

phase space volume remains constant.