Participation Project: Lecture Notes on Central Force Problem (Planetary Motion) Submitted to Professor Dr.Sebastian Kuhn

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February 2020

1 Important Terms

(a)Reduced mass(μ) and Kinetic energy(T) of equivalent one body system of two masses m_1 and m_2 are :

$$\mu = \frac{m_1.m_2}{m_1 + m_2}$$
$$T = \frac{1}{2}\mu\dot{r}^2 + \frac{p_\phi^2}{2\mu r^2}$$

(b)Gravitational Potential energy (as a function of position only) is given by

$$V = V(r) = \frac{-Gm_1m_2}{r}$$

2 General Approach to Central Force problem and Kepler's laws of planetary motion:

Total energy of a two-body system is:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + V(r)$$

Since

$$\dot{\phi} = \frac{p_\phi}{\mu r^2} \tag{1}$$

$$E = \frac{1}{2}\mu \dot{r^2} + \frac{1}{2\mu r^2}p_\phi^2 + V(r) \tag{2}$$

$$E = \frac{1}{2}\mu \dot{r^2} + \frac{1}{2\mu r^2}p_\phi^2 - \frac{G\mu M}{r} \tag{3}$$

$$\vec{r} = r(\vec{\phi}) \tag{4}$$

$$\dot{r} = \frac{\partial r}{\partial \phi}\dot{\phi} = \frac{\partial r}{\partial \phi}\frac{p_\phi}{\mu r^2} \tag{5}$$

Lets introduce a new variable $u(\phi)$ defined as:

$$u(\phi) = \frac{1}{r(\phi)} \implies \frac{\partial u}{\partial \phi} = \frac{-1}{r^2}\frac{\partial r}{\partial \phi} \tag{6}$$

$$\implies \dot{r} = -r^2 \frac{du}{d\phi}\frac{p_\phi}{\mu r^2} = \frac{-p_\phi}{\mu}\frac{du}{d\phi} = \frac{-p_\phi}{\mu}\dot{u} \tag{7}$$

where $\dot{u} = \frac{\partial u}{\partial \phi}$.

So, equation (3) can be written as:

$$E = \frac{1}{2}\mu \frac{p_\phi^2}{\mu^2}\dot{u}^2 + \frac{p_\phi^2}{2\mu}u^2 - G\mu M u \tag{8}$$

$$\dot{u}^2 + u^2 - \frac{2G\mu^2 M}{p_\phi^2} = \frac{2\mu E}{p_\phi^2} \tag{9}$$

$$\dot{u}^2 = \frac{2\mu.E}{p_\phi^2} + \frac{2G\mu^2 M}{p_\phi^2}u - u^2 \tag{10}$$

$$\dot{u} = \sqrt{\frac{2\mu.E}{p_\phi^2} + \frac{2G\mu^2 M}{p_\phi^2}u - u^2} \tag{11}$$

$$\tag{12}$$

For maximum or minimum value of r (i.e at equilibrium)

$$\dot{r} = 0 \implies \frac{\partial u}{\partial \phi} = 0$$

$$u^2 - \frac{2G\mu^2 M}{p_\phi^2}u - \frac{2\mu.E}{p_\phi^2} = 0$$

$$u = \frac{1}{2}[\frac{2G\mu^2 M}{p_\phi^2} \pm \sqrt{4 \times \frac{G^2\mu^4 M^2}{p_\phi^4} - 4 \times 1 \times \frac{-2\mu E}{p_\phi^2}}]$$

$$u = \frac{G\mu^2 M}{p_\phi^2} \pm \sqrt{\frac{G^2\mu^4 M^2}{p_\phi^4} + \frac{2\mu E}{p_\phi^2}}$$

$$\implies u_{max} = \frac{G\mu^2 M}{p_\phi^2} + \sqrt{\frac{G^2\mu^4 M^2}{p_\phi^4} + \frac{2\mu E}{p_\phi^2}} \implies r_{min}$$

$$u_{min} = \frac{G\mu^2 M}{p_\phi^2} - \sqrt{\frac{G^2\mu^4 M^2}{p_\phi^4} + \frac{2\mu E}{p_\phi^2}} \implies r_{max}$$

Integrating left side from u_{min} to u_{max} so that ϕ goes from $\phi_{umin}$to ϕ_{umax}on right side,

$$\int_{u_{min}}^{u_{max}} \frac{du}{\sqrt{\frac{2\mu.E}{p_\phi^2} + \frac{2G\mu^2 M}{p_\phi^2}u - u^2}} = \int_{\phi_{min}}^{\phi_{max}} d\phi = \Phi_{max} - \Phi_{min} \qquad (13)$$

$$(14)$$

Now we define a new variable v as:

$$v = u - \frac{G\mu^2 M}{p_\phi^2} \implies dv = du \implies v^2 = u^2 - \frac{2G\mu^2 M}{p_\phi^2}u + \frac{G^2\mu^4 M^2}{p_\phi^4} \qquad (15)$$

Using (14) in (13), we get:

$$\Phi_{max} - \Phi_{min} = \int_{v_{min}}^{v_{max}} \frac{dv}{\sqrt{\frac{2\mu E}{p_\phi^2} + \frac{G^2\mu^4 M^2}{p_\phi^4} - v^2}} = \int \frac{dv}{\sqrt{A^2 - v^2}}$$

$$A^2 = \frac{2\mu E}{p_\phi^2} + \frac{G^2\mu^4 M^2}{p_\phi^4}$$

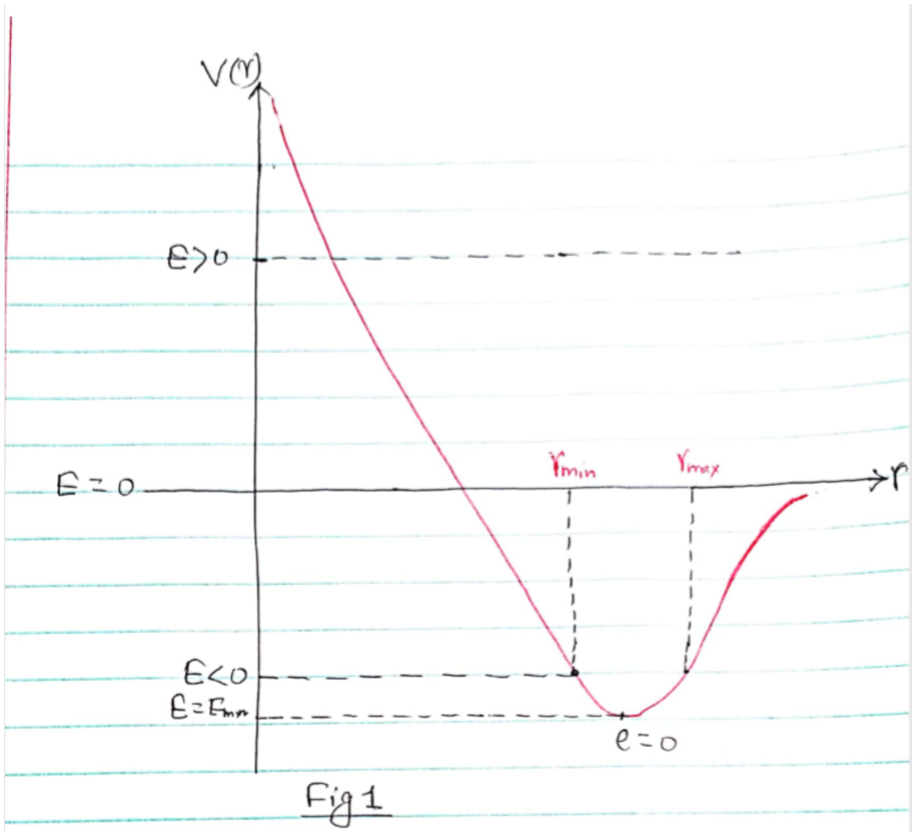

Figure 1: variation of potential with distance in central force problem: this gives the condition for various types of conic section.

Where,

$$v_{min} = u_{min} - \frac{G\mu^2 M}{p_\phi^2} = -A$$

$$v_{max} = u_{max} - \frac{G\mu^2 M}{p_\phi^2} = +A$$

$$\implies \Phi_{u_{max}} - \Phi(u) = \int_{u-\frac{G\mu^2 M}{p_\phi^2}}^{A} \frac{dv}{\sqrt{A^2 - v^2}} = \int_{\omega_0}^{1} \frac{d\omega}{\sqrt{1-\omega^2}}$$

$$= |\arccos(\omega)|_{\omega_0}^{1} = -\arccos(\omega_0)$$

Where $\omega = \frac{v}{A} \implies dv = Adw$ and $\omega_0 = (u - \frac{G\mu^2 M}{p_\phi^2})/A$.

Lets define

$$\phi_{umax} = 0$$
$$and, \phi(u) = \phi$$
$$\implies \arccos \omega_0 = \phi$$
$$\arccos w_0 = \phi \implies w_0 = \cos\phi$$
$$\implies \cos\phi = \frac{1}{A}[u - \frac{G\mu^2 M}{p_\phi^2}]$$
$$u - \frac{G\mu^2 M}{p_\phi^2} = A\cos\phi$$
$$u = u(\phi) = A\cos\phi + \frac{G\mu^2 M}{p_\phi^2}$$
$$\frac{1}{r} = \frac{G\mu^2 M}{p_\phi^2}(1 + e\cos\phi)$$
$$\frac{1}{r} = c(1 + e\cos\phi)$$

Where

$$c = \frac{G\mu^2 M}{p_\phi^2} \tag{16}$$

$$e = \frac{A}{\frac{G\mu^2 M}{p_\phi^2}} = \sqrt{\frac{2Ep_\phi^2}{G^2\mu^3 M^2} + 1} \tag{17}$$

Where 'e' is called eccentricity.The different values of e gives different shapes of the orbit as:

$e = 0 \implies E = -\frac{G^2\mu^3 M^2}{2p_\phi^2}$ (circular orbit),
$e < 1 \implies E < 0$; ellipse
$e = 1 \implies E = 0$; parabola
$e > 1 \implies E > 0$; Hyperbola

$$r(\phi) = \frac{1}{c(1 + e\cos\phi)}$$
$$\implies r_{min} = a(1 - e)$$
$$r_{max} = a(1 + e)$$
$$X(\phi) = r(\phi)\cos\phi$$
$$Y(\phi) = r(\phi)\sin\phi$$
$$Y_{max} = b = a\sqrt{1 - e^2}$$

since, $e = 1$ for parabola

$$e = 1 \implies \frac{2Ep_\phi^2}{G^2\mu^3M^2} = 1 - e^2 = 0$$
$$\implies E = 0$$

Case (II) $e < 1$ Ellipse:

$$e^2 = \frac{2Ep_\phi^2}{G^2\mu^3M^2} + 1$$
$$1 - e^2 = \frac{-2Ep_\phi^2}{G^2\mu^3M^2}$$
$$1 - e^2 = +ve \implies E = \frac{-(1-e^2)G^2\mu^3M^2}{2p_\phi^2} = -ve$$

this system is a bound system(-ve energy)

$$r(\phi) = \frac{a(1-e^2)}{1 + e\cos\phi}$$
$$\implies \frac{1}{c} = a(1-e^2)$$
$$\implies a = \frac{1}{c} \times \frac{1}{1-e^2}$$
$$= \frac{p_\phi^2}{G\mu^2M} \times \frac{G\mu^3M^2}{(-2E)p_\phi^2}$$
$$\implies a = \frac{-G\mu M}{2E}$$

And,

$$b = a\sqrt{1-e^2} = \frac{-G\mu M}{2E} \times \sqrt{\frac{-2Ep_\phi^2}{G^2\mu^3M^2}}$$
$$= p_\phi\sqrt{\frac{1}{(-2E)\mu}}$$

Where 'a'is called Semi-major axis of the ellipse and 'b' is called semi-minor axis of the ellipse.

Now, area of the ellipse is given by:

$$A = \pi \times a \times b$$
$$= \pi(\frac{-G\mu M}{2E}) \times p_\phi \times \sqrt{\frac{1}{(-2E)\mu}}$$

Now, the time period is given by:

$$
\begin{aligned}
T &= \frac{area}{arealvelocity} \\
&= \frac{\pi ab}{\frac{1}{2}\frac{p_\phi}{\mu}} \\
&= \frac{-2\pi \times G\mu M}{2E} \times p_\phi \sqrt{\frac{1}{(-2E)\mu}} \times \frac{1}{\frac{p_\phi}{\mu}} \\
T &= 2\pi \sqrt{\frac{G^2\mu^3 M^2}{(-2E)^3}} \\
T &= 2\pi \frac{1}{\sqrt{GM}} a^{\frac{3}{2}} \implies T^2 \alpha a^3
\end{aligned}
$$

Thus , the square of time period of elliptical orbits(eg. orbits of Planets) is directly proportional to the cube of the semi major axis of the ellipse. This is called Kepler's third law of planetary motion.