1. Independent Coordinates of a Rigid Body

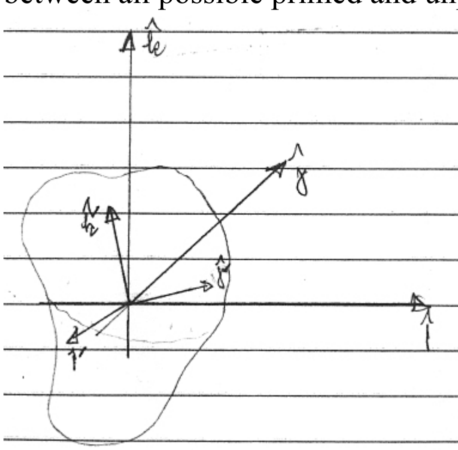
Object made of 10^{23} atoms/molecules seems to have $N=3\cdot10^{23}$ degrees of freedom. However, as already discussed in 2nd lecture, there are N-6 constraints since the distance between any two atoms/molecules is fixed. Hence, the position of every atom/molecule can be given by 6 generalized variables: the position of one (arbitrary) atom/molecule (3 d.o.f.), the relative position of a 2nd one (2 d.o.f., e..g 2 angles, since the distance to the first one is fixed) and the direction of a 3rd one NOT on the same straight line as the first two (1 d.o.f., since its distance to both initial two is fixed; this d.o.f. can be an angle around the axis connecting the first two). All other atoms/molecules have their position fixed by the requirement of a fixed distance from the 3 first ones.

These 6 d.o.f. can be chosen in the following way:

a) Pick one point fixed relative to the rigid body. This can be its center of mass, or any atom/molecule inside, or even a "virtual point" that is specified by its position relative to the body. This point can be described by the usual 3 Cartesian coordinates (x,y,z).

b) Choose a second set of coordinates (x',y',z') with origin at the point chosen under a) and convenient, fixed orientation relative to the rigid body. By describing the orientation of the primed coordinate system (which will require 3 generalized coordinates, see below), we can fix the orientation of the body in space.

In the following, we ignore the coordinates of the fixed point (a), i.e. we are assuming that both origins of the primed and unprimed coordinate systems coincide. (See later). The unprimed coordinate system is defined by the unit vectors $\{\hat{i},\hat{j},\hat{k}\}$ in {x,y,z} direction, respectively, and the primed coordinate system similarly by $\{\hat{i}',\hat{j}',\hat{k}'\}$. **One** way to express the orientation of the rigid body is then by giving the direction cosines between all possible primed and unprimed axes:



$$\hat{i}'\cdot\hat{i} = \cos\theta_{11}$$
$$\hat{i}'\cdot\hat{j} = \cos\theta_{12}$$
$$\hat{i}'\cdot\hat{k} = \cos\theta_{13}$$
$$\hat{j}'\cdot\hat{i} = \cos\theta_{21}$$
$$\hat{j}'\cdot\hat{j} = \cos\theta_{22}$$
$$\hat{j}'\cdot\hat{k} = \cos\theta_{23}$$
$$\hat{k}'\cdot\hat{i} = \cos\theta_{31}$$
$$\hat{k}'\cdot\hat{j} = \cos\theta_{32}$$
$$\hat{k}'\cdot\hat{k} = \cos\theta_{33}$$

This is certainly overkill, since now we have 9 "coordinates" instead of just 3. We will see later how to express these 9 in terms of 3 judiciously chosen ones (there is more than one way to do that). We can organize these direction cosines into a matrix

$$\mathbf{R} = \begin{pmatrix} \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33} \end{pmatrix} \text{ with } R_{lm} = \cos\theta_{lm} \text{ and } \hat{u}'_l = \sum_{m=1}^{3} R_{lm}\hat{u}_m \text{, where } \hat{u}_1 = \hat{i} \text{ etc.}$$

We designate with r_m the coordinates of a vector **r** in the unprimed coordinate system , and with r_m' the coordinates of the **same** vector in the primed coordinate system. Hence, $r_l' = \vec{\mathbf{r}} \cdot \hat{u}'_l = \vec{\mathbf{r}} \cdot \sum_{m=1}^{3} R_{lm}\hat{u}_m = \sum_{m=1}^{3} R_{lm}\vec{\mathbf{r}} \cdot \hat{u}_m = \sum_{m=1}^{3} R_{lm}r_m$, i.e. the **components** of the vector transform following the usual rules of matrix multiplication: $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{R}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The matrix **R** thus describes a **passive** rotation (where the vector remains unchanged but its coordinates are different in the rotated coordinate system). The same is true for the (components of) **any** vector – in fact, we **define** a vector as a quantity with 3 components that transform according to those same rules of matrix multiplication under a rotation of the coordinate system.

2. Orthogonal Transformations

Finally, because the unit vectors describing the axes in the primed coordinate system are normalized to one and are orthogonal to each other, we have

$$\hat{u}'_l \cdot \hat{u}'_n = \sum_{m=1}^{3} R_{lm}\hat{u}_m \cdot \sum_{m'=1}^{3} R_{nm'}\hat{u}_{m'} = \sum_{m=1}^{3} R_{lm}R_{nm} = \sum_{m=1}^{3} \cos\theta_{lm}\cos\theta_{nm} = \delta_{\ln}$$

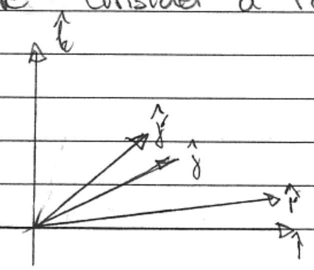
which yields 6 independent equations (3 for equal indices and 3 for the possible combinations of unequal ones). This in turn explains how the 9 direction cosines must depend on only 3 independent parameters. We call any matrix that fulfills the above equation orthogonal.

The relationship shown above can also be gotten by observing that the scalar product between any two vectors must be the same in the primed and unprimed coordinate system since the physical vectors don't change. In matrix multiplication language,

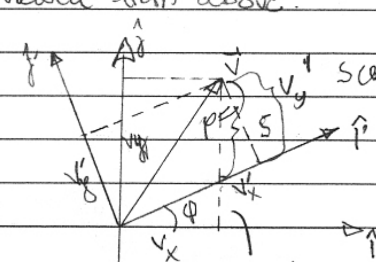
$$\vec{\mathbf{v}}\cdot\vec{\mathbf{u}} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} = (\mathbf{v})^T(\mathbf{u})$$

Using the fact that $(\mathbf{M}\,\mathbf{V})^{\mathbf{T}} = (\mathbf{V})^{\mathbf{T}}(\mathbf{M})^{\mathbf{T}}$ for any two matrices, with $(\mathbf{M})^{\mathbf{T}}{}_{lm} = \mathbf{M}_{ml}$, we can see that $(\mathbf{v}')^T(\mathbf{u}') = (\mathbf{v})^T(\mathbf{R})^T(\mathbf{R})(\mathbf{u})$ for any arbitrary vectors **v**, **u**. Hence the matrix **R** must fulfill the equation $(\mathbf{R})^{\mathbf{T}}(\mathbf{R}) = \mathbf{1}$, i.e the transpose of the rotation matrix must be its inverse.

Example: Consider a rotation about the z-axis ($\hat{z}$ fixed)



Viewed from above:



$s\cos\varphi = V_y'$

$V_x' \tan\varphi + s = V_y$

$$\mathbf{R} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Bigg\} \text{ Passive}$$

$V_y' = (V_y - V_x \tan\varphi)\cos\varphi$

$= V_y\cos\varphi - V_x\sin\varphi$

Use this example to check all of the abstract relationships above.
Note (WARNING! CONFUSING!): we can also use the same rotational matrix to express an **active** rotation where an initial vector **v** is turned into a vector **u** that is rotated around the z-axis by an angle $-\phi$. Since this $\mathbf{R}(-\phi)$ must be equal to $\mathbf{R}^{-1}(\phi) = \mathbf{R}^{T}(\phi)$ we can see that the components of the rotated vector **u** must be given by $(\mathbf{u})_l = (\mathbf{R})^{T}{}_{lm}(\mathbf{V})_m = (\mathbf{R})_{ml}(\mathbf{V})_m$ (this is generally true: the matrix describing the coordinate transformation of a vector under an **active** rotation is equal to the transpose of the matrix describing the coordinate transformation of an unchanged vector under a passive rotation of the same kind.

3. Properties of Matrices

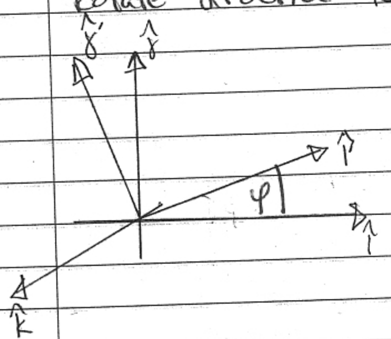
ONLY as needed: definition of matrix multiplication, associative law, vectors as column matrices, transpose vectors as row matrices, inverse matrix, unit matrix, transpose matrix, "square" = symmetric matrix, determinant and multiplication properties, determinant of transpose, trace, proper rotations (with det = 1), reflection, etc. (see chapter 4.3-4.4 in Goldstein).

4. Euler Angles

This is one standard method to define the orientation of the rigid body-fixed primed coordinate system relative to the (stationary) unprimed system. Basically, two of the Euler angles just describe the orientation of the z’ axis in the unprimed coordinate system using polar and azimuthal angles, θ and ϕ. The final angle, y, describes the rotation of the body around its own z’ axis, i.e. the orientation of x’ and y’. Since rotations generally **do not commute**, some care has to be taken to define this precisely – namely as a sequence of rotations:

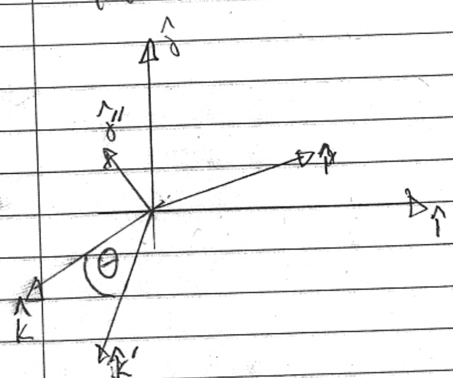
Method 2

Rotate around $\hat{k}$



$$\bar{R}_1 = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

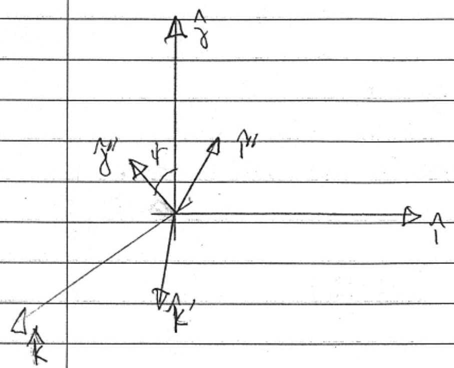
Rotate around $\hat{i}'$



$$\bar{R}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$(\rightarrow)$

Rotate around $\hat{k}'$



$$\bar{R}_3 = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, $\tilde{R} = \bar{R}_3 \bar{R}_2 \bar{R}_1$

Note the “reversed” order in the product of the 3 rotations, since by definition the rightmost matrix gets applied first on any vector the product acts on, then the 2nd etc. More details and pictures in Goldstein. The explicit form of the matrix for a general rotation with 3 Euler angles is also left for the students to figure out in HW 5.
The end effect is that the 3 angles introduced here completely describe the orientation of a rigid body, once a fixed point (with 3 coordinates) has been chosen. Together with those 3 coordinates, they form a system of 6 generalized coordinates that automatically fulfill all constraints.

5. (Skip)

6. Euler’s Theorem on the Motion of a Rigid Body

Alternative way to express a rotation of the body-fixed primed coordinate system (and hence the body itself): Euler’s Theorem states that **any** rotation can be expressed as a single rotation around a fixed axis $\hat{n}$ with finite angle ϕ: Given an arbitrary rotational matrix **R**, find a vector **n** such that its components are invariant (unchanged) under the rotation. Thus we can describe the full rotation as a sequence of transformation where first we transform into a coordinate system where the z’’ axis is along the vector **n**, then rotate around **n** (rotation around the z’’’ axis analog to the Example in 2.) and then transform back into the original coordinate system. Only the middle step involves an active rotation, and hence the full rotation is equivalent to the rotation around **n**.
We are left with finding the vector **n** such that (**n**) = (**R**)(**n**) in the usual sense of matrix multiplication. This is a special case of the general eigenvector/eigenvalue problem

$(\mathbf{R})(\vec{\mathbf{A}}) = \lambda(\vec{\mathbf{A}})$ which is familiar from Quantum Mechanics. The details are in Chapter 4.6 in Goldstein, but here is the outline:

Consider rotation $\bar{R}$

$$\bar{R} \cdot (\vec{A}) = \lambda \vec{A}$$

so

$$(\bar{R} - \lambda \bar{I}) \vec{A} = 0,$$

hence

$$\det(\bar{R} - \lambda \bar{A}) = 0$$

$a\lambda^3 + b\lambda^2 + c\lambda + d = 0$; cubic has 3 solutions, $\lambda_1, \lambda_2, \lambda_3$ which correspond to $\vec{A}_1, \vec{A}_2, \vec{A}_3$. Since these are vectors, construct the matrix

$$\bar{A} = (\vec{A}_1\ \vec{A}_2\ \vec{A}_3)$$

where $\vec{A}_1, \vec{A}_2, \vec{A}_3$ are column vectors. Now

$$\bar{A}^{-1}\bar{R}\bar{A} = A^{-1}(\lambda_1\vec{A}_1, \lambda_2\vec{A}_2, \lambda_3\vec{A}_3)$$

$$= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Hence

$$\lambda_1\lambda_2\lambda_3 = 1 \quad \text{since } \det(\bar{A}^{-1}\bar{R}\vec{A}) = \det(A^{-1})\,1\,\det(A) = 1$$

This assumes that we can find three eigenvectors $\mathbf{A}_i$ for the three eigenvalues λ_i that are linearly independent (which we know from QM is true) and hence can form a matrix $\mathbf{A}$ as shown above which has an inverse.

Here is a trick to show that at least one of the λ's must be = 1 (let's call it λ_1):

$$\det((\bar{R} - \bar{I})\bar{R}^T) = \det(\bar{I} - \bar{R}^T)$$

$$= \det(\bar{I} - \bar{R})$$

$$\det(R - I)\det(R^T) = \det\left[(\bar{R} - \bar{I}) \cdot \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$= \det(\bar{R} - \bar{I}) \cdot \det\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\det(R - I) \cdot 1$$

$$= (-1)\det(\bar{R} - \bar{I}) \Rightarrow \det(R - I) = 0$$

The other two λ's must multiply to 1 since the product of all three is 1. Hence we can write them as $\lambda_2 = \lambda_3^* = e^{i\varphi}$. Either all three of them are equal to 1 (ϕ =0) in which case the whole theorem becomes trivial (**R** is the unit matrix and hence can be described by a rotation around any axis with angle 0), or the last two are both equal to -1 ($\phi = \pi$), which corresponds to a rotation around 180 degrees (leading to a simple sign change of the coordinates perpendicular to $\mathbf{A}_1 = \mathbf{n}$) or they are genuinely complex, in which case the corresponding eigenvectors must be complex, as well, and hence the axis of rotation is uniquely determined. Furthermore, the matrix in the double-primed coordinate system simply looks like $\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so we have both **n** and ϕ. We can also get ϕ directly by keeping in mind that the trace = 1 + 2 cosϕ is unchanged going to the double-primed coordinate system and hence we can just calculate the trace of the original matrix **R**.

7. Finite Rotations

We now want to express the rotation around the axis **n** with an angle ϕ independent of coordinate systems. This can be done by observing that not only the scalar product, but also the vector product between two vectors is preserved by coordinate transformations. In other words, the vector **uxv** is the same physical vector, regardless what coordinate system we use to describe its coordinates as well as those of **u** and **v**. (It's tedious but straightforward to show this directly by calculating the coordinates in the primed coordinates system of Section 1 using the matrix **R**.)

Going into the double-primed coordinate system where **n** is along the z'' axis, we can look at some arbitrary vector **v** as having two components: One, $\vec{\mathbf{v}}_{\parallel} = \left(\vec{\mathbf{v}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$ along the direction of the axis of rotation (z'' direction) and remains unchanged, and the other one, $\vec{\mathbf{v}}_{\perp} = \vec{\mathbf{v}} - \vec{\mathbf{v}}_{\parallel}$ perpendicular to it. We might as well pick our x'' axis along the direction of this perpendicular component: $\hat{i}'' = \vec{\mathbf{v}}_{\perp}\big/\left|\vec{\mathbf{v}}_{\perp}\right|$, so that **v** has no y''-component. This third axis (y'') is then given by the direction of $\hat{\mathbf{n}}\times\vec{\mathbf{v}} = \hat{\mathbf{n}}\times\vec{\mathbf{v}}_{\perp}$ (which has obviously the same magnitude as $\vec{\mathbf{v}}_{\perp}$, as a vector product of two perpendicular vectors, one of which has unit length): $\hat{j}'' = \hat{\mathbf{n}}\times\vec{\mathbf{v}}\big/\left|\vec{\mathbf{v}}_{\perp}\right|$. The rotation designated by the matrix above describes either a passive rotation by the angle ϕ in counterclockwise direction or (the position taken here) an **active** rotation of the vector **v** in clockwise direction (both in the sense that we are looking down along **n** towards the origin), i.e. $\vec{\mathbf{v}}_{\perp}$ is rotated towards $-\hat{\mathbf{n}}\times\vec{\mathbf{v}} = \vec{\mathbf{v}}\times\hat{\mathbf{n}}$. The rotated vector can thus be constructed as

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}}_{\parallel} + \cos\phi\left|\vec{\mathbf{v}}_{\perp}\right|\hat{i}'' - \sin\phi\left|\hat{\mathbf{n}}\times\vec{\mathbf{v}}_{\perp}\right|\hat{j}'' = \vec{\mathbf{v}}_{\parallel} + \cos\phi\,\vec{\mathbf{v}}_{\perp} - \sin\phi\left(\hat{\mathbf{n}}\times\vec{\mathbf{v}}\right)$$

$$= \left(\vec{\mathbf{v}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \cos\phi\left(\vec{\mathbf{v}} - \left(\vec{\mathbf{v}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}\right) + \sin\phi\left(\vec{\mathbf{v}}\times\hat{\mathbf{n}}\right) = \vec{\mathbf{v}}\cos\phi + \left(\vec{\mathbf{v}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}(1-\cos\phi) + \left(\vec{\mathbf{v}}\times\hat{\mathbf{n}}\right)\sin\phi$$

(see also the figures Goldstein for further explanation). If we want to describe an **active** rotation in **counter-clockwise** direction, we simply have to change the sign of ϕ and therefore get $\vec{\mathbf{v}}' = \vec{\mathbf{v}}\cos\phi + (\vec{\mathbf{v}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}(1-\cos\phi) + (\hat{\mathbf{n}}\times\vec{\mathbf{v}})\sin\phi$. From now on, we will assume that all rotations are counter-clockwise, i.e. the axis **n** forms the "thumb" of a right-hand "screw" where the fingers indicate the direction of rotation for positive ϕ.

8. Infinitesimal Rotations

Looking at the last equation above, we can see that it simplifies for the case of a rotation around some axis **n** by an infinitesimally small angle $d\Omega$:
$d\vec{\mathbf{v}} = \vec{\mathbf{v}}' - \vec{\mathbf{v}} \approx (\hat{\mathbf{n}}\times\vec{\mathbf{v}})\,d\Omega =: d\vec{\mathbf{\Omega}}\times\vec{\mathbf{v}}$ with $d\vec{\mathbf{\Omega}} = \hat{\mathbf{n}}\,d\Omega$. Here, $d\vec{\mathbf{\Omega}}$ behaves like an ordinary vector under coordinate transformations, but it does not change sign under space inversions (parity transformation). Hence, it is an **axial vector**.

In matrix formulation, we can write infinitesimal rotations as $\mathbf{R} = \mathbf{1} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon}$ is a "small" matrix close to zero. In the following, we will neglect all terms that are quadratic or of higher order in the $\boldsymbol{\varepsilon}$'s. We can write a rotation $d\vec{\mathbf{\Omega}}$ around some axis $\hat{\mathbf{n}}$ by breaking it down to its components $(\vec{\mathbf{\Omega}})_i$ along each of the coordinate axes. This is possible for infinitesimal rotations (only!) because they commute:

$\vec{r}\,' = (\bar{I} + \bar{\varepsilon})\vec{r}$, where $\bar{R} = (\bar{I} + \bar{\varepsilon})$

Infinitesimal rotations commute; i.e.,

$$\bar{R}_1\bar{R}_2\bar{R}_3 = (\bar{I}+\bar{\varepsilon}_1)(\bar{I}+\bar{\varepsilon}_2)(\bar{I}+\bar{\varepsilon}_3)$$
$$= \bar{I} + \bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \dots$$

which, in the limit goes to $\bar{I}$. Hence, $(\mathbb{1}+\bar{\varepsilon})(\mathbb{1}+\bar{\varepsilon}^T) \stackrel{!}{=} \mathbb{1}$

$$\Rightarrow \bar{\varepsilon}^T = -\bar{\varepsilon}$$

Which implies $\bar{\varepsilon}$ looks something like this

$$\bar{\varepsilon} \sim \begin{pmatrix} 0 & -d\Omega_3 & d\Omega_2 \\ d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & d\Omega_1 & 0 \end{pmatrix}$$

($\rightarrow$)

(Use a book or some other rectangular object to demonstrate that, in general, rotations to **not** commute, see Figs.4.9-4.10 in Goldstein).

We can introduce a "vector of matrices" in the following way:

$$\vec{\mathbf{M}} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

. These can be called "generators of rotations" and are represented by the components of the angular momentum operator in Quantum Mechanics. They form a "Lie Algebra" for the group SO(3) of rotations with commutation relationship $\mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i = \left[\mathbf{M}_i\mathbf{M}_j\right] = \sum_k \varepsilon_{ijk}\mathbf{M}_k$:

$d\Omega_3 = \hat{n}_3 \cdot d\Phi$, $d\Omega_2 = (\hat{n})_2 d\Phi$, $d\Omega_1 = (\hat{n})_1 d\Phi$

$\vec{r}' = \vec{r} + (\vec{r} \times \hat{n}) d\Phi$ $(\cos d\phi \simeq 1)$

$$= \vec{r} + \left[(\hat{n})_1 d\Phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + (\hat{n})_2 d\Phi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + (\hat{n})_3 d\Phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \vec{r}$$

$$= \vec{r} + \left((\hat{n})_1 d\Phi\, M_1 + (\hat{n})_2 d\Phi\, M_2 + (\hat{n})_3 d\Phi\, M_3 \right) \vec{r}$$

For active rotation COUNTER clockwise

M_1, M_2, M_3 = generators of rotations, Lie Algebra for SO(3)

$$\vec{r}' = (\mathbb{1} + \hat{n} \cdot \vec{M}\, d\phi)(\vec{r}) = (\mathbb{1} + \vec{d\phi} \cdot \vec{M})$$

9. Rate of Change of a Vector

Picking up at the beginning of 8. above, we can now describe the motion of **any** vector **v** with time as composed of 2 parts: (1) motion relative to the body-fixed set of coordinates (primed) plus (2) rotation of the whole system (body with primed coordinates). For an infinitesimally short time interval, these two changes commute and we can write $\left(d\vec{\mathbf{v}}\right)_S = \left(d\vec{\mathbf{v}}\right)_{S'} + \left(d\vec{\mathbf{v}}\right)_{rot} = \left(d\vec{\mathbf{v}}\right)_{S'} + d\vec{\mathbf{\Omega}} \times \vec{\mathbf{v}}$, where the last term contains the infinitesimal rotation d**Ω** of the body during the time *dt*. (Remember that the we separated out any center-of-mass motion so that any displacement of the body can be described as a rotation around a single specific point, around some axis going through that point). The subscript S indicates the change as measured in the fixed space coordinate system (unprimed), while the subscript S' indicates the change relative to the body-fixed (co-rotating, primed) coordinate system.

We can divide by the time dt elapsed and get $\left(\frac{d\vec{\mathbf{v}}}{dt}\right)_S = \left(\frac{d\vec{\mathbf{v}}}{dt}\right)_{S'} + \frac{d\vec{\mathbf{\Omega}}}{dt} \times \vec{\mathbf{v}} = \left(\frac{d\vec{\mathbf{v}}}{dt}\right)_{S'} + \vec{\omega} \times \vec{\mathbf{v}}$.

The vector **ω** is the angular velocity vector that defines both the speed of rotation and the axis around which the rotation occurs. It is of course independent of any coordinate system, but its **components** depend on the coordinate system. In the future, it will be most convenient to write down the components of **ω** in the primed (body-fixed) coordinate system. We can express them in terms of the Euler angles in the following way. Recall that the Euler angles are applied in the following order: First a rotation by ϕ around the z-axis, then a rotation by θ around the x' axis, then a rotation by ψ around the z' axis. Hence,

1) A change in ψ has only a z' component, by definition
2) A change in θ affects both the x' and y' components, depending on their orientation. If ψ=0, a change in θ equals a rotation around x', but in general there are components in both x' (proportional to $\cos\psi$) and in y' (proportional to $-\sin\psi$) directions.
3) Finally, a change in ϕ affects all three components. If ψ=0, there is no component along the x' axis, and all we have is the projection on the z' axis (proportional to $\cos\theta$) and on the y' axis (proportional to $\sin\theta$). For arbitrary ψ, the second component gets further split up in y'-direction (proportional to $\cos\psi$) and x'-direction (proportional to $\sin\psi$).

You can use your right hand to try and convince yourself of these relationships, but I am not liable for any injuries you might sustain! Collecting all terms for each axis, we find $\left(\vec{\omega}\right)_{z'} = \dot{\psi} + \dot{\phi}\cos\theta, \left(\vec{\omega}\right)_{x'} = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi, \left(\vec{\omega}\right)_{y'} = \dot{\theta}(-\sin\psi) + \dot{\phi}\sin\theta\cos\psi$.

You could also find this by first expressing the vector **ω** in the unprimed system (which is not that much simpler) and then applying the transformation matrix from the unprimed to the primed coordinate system (see HW. 5).

10. Coriolis Effect

As time permits… If we use **ω** to describe Earth's rotation around its axis, we can use the equation in 9. to write the motion of an object relative to "absolute space" (S) and relative to the coordinate system fixed on Earth (S'): $\left(\frac{d\vec{r}}{dt}\right)_S = \left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega} \times \vec{r}$.

The second derivative gives

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_S = \frac{d}{dt}\left(\left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}\times\vec{r}\right)_{S'} + \vec{\omega}\times\left(\left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}\times\vec{r}\right) = \left(\frac{d^2\vec{r}}{dt^2}\right)_{S'} + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}\times\vec{\omega}\times\vec{r}$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_{S'} + 2\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}\left(\vec{\omega}\cdot\vec{r}\right) - \omega^2\vec{r}$$

From the point of the Earth-fixed (primed) coordinate system, it thus looks as if there are two extra (pseudo-)forces acting on any moving object:

$$\vec{F}_{S'} = \vec{F}_S - 2m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{S'} + m\left(\omega^2\vec{r} - \vec{\omega}\left(\vec{\omega}\cdot\vec{r}\right)\right).$$

The second term is the so-called Coriolis force and the last term is the (generalized) "centrifugal force". The latter leads to a radially outward pointing force (counteracting gravity) plus a force towards the equatorial plane for any position at higher (positive or negative) latitudes. Of course, at the poles these two forces cancel which is not surprising, as the rotation does not actually contribute to any motion there. The Coriolis force gives a push "sideways" to an object moving in any direction other than Earth's axis. In particular, an object moving on Earth's surface at any point other than the Equator will have a component of its velocity perpendicular to ω and thus be pushed East if it moves North on the Northern Hemisphere, North if it moves West etc. This means that missiles will be missing the mark they are aimed at, and a stream of fluid (air, ocean water, etc.) will be bend into a circular pattern (counterclockwise around a low pressure system[1] in the Northern Hemisphere, clockwise in the southern one). Hence the regular patterns of hurricanes, ocean currents etc.

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[1] Imagine a region of low pressure in a middle latitude (45 degrees North). Air will be rushing towards this from the South – getting diverted into an easterly direction, and from the North – getting diverted into a westerly direction etc. The net effect is a counter-clockwise motion around that center of low pressure – a hurricane.