Small Oscillation

In the following, we discuss small oscillation in the motion of a system within the immediate neighborhood of a configuration of stable equilibrium. However, it's important to point out that the same methods also apply for larger oscillations if the Lagrangian can be cast in the same form as spelled out below.

The system is said to be in equilibrium when the generalized forces (Q_i) acting on the system vanish:

$$Q_i = -\frac{\partial V}{\partial q_i} = 0 \qquad 1$$

If a small disturbance of the system from equilibrium results only in small oscillatory motion about the rest position, such equilibrium is classified as stable. In the small bound motion, if the departures from the equilibrium are small, all functions may be expanded in a Taylor series about the equilibrium retaining only the lowest-order terms. The deviation of the generalized coordinates (q_i) from equilibrium position (q_{i0}) is denoted by η_i.

That is, $$q_i = q_{i0} + \eta_i \qquad 2$$

which implies that,

$$\eta_i = q_i - q_{i0}$$
$$\dot{\eta}_j = \frac{d}{dt}\left(q_j - q_{j0}\right) = \dot{q}_j$$

Then,

$$V(q_i) = V(q_{i0} + \eta_i)$$

Using Taylor series expansion:

$$V(q_i) = \sum_i V(q_{i0}) + \sum_i (\frac{\partial V}{\partial q_i})\eta_i + \frac{1}{2}\sum_{ij}(\frac{\partial^2 V}{\partial q_i \partial q_j})\eta_i \eta_j + \ldots\ldots.. \qquad 3$$

By shifting the arbitrary zero of potential to coincide with the equilibrium potential, the first term in (3) must vanish. Using (1), the second term (linear in η_i) vanishes.

Then,

$$V(q_i) = \frac{1}{2}\sum_{ij} V_{ij}\eta_i\eta_j \tag{4}$$

Where $V_{ij} = (\frac{\partial^2 V}{\partial q_i \partial q_j}) = V_{ji}$ (Symmetric)

Since the generalized coordinates do not involve the time explicitly; the kinetic energy is a homogenous quadratic function of the velocities and it is given by

$$T = \frac{1}{2}\sum_{ij} T_{ij}\dot{\eta}_i\dot{\eta}_j \tag{5}$$

Where $T_{ij} = M_{ij} = T_{ji}$ (Symmetric)

Now, the Lagrangian is given by

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}\sum_{ij} T_{ij}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2}\sum_{ij} V_{ij}\eta_i\eta_j \end{aligned} \tag{6}$$

Taking η_i as the generalized coordinates, the Lagrangian leads to the following n-equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} = 0$$

$$\sum_{ij} T_{ij}\ddot{\eta}_j + \sum_{ij} V_{ij}\eta_{ij} = 0 \tag{7}$$

Solution:

The equations of motion (7) are linear differential equations with constant coefficients and its oscillatory solution is given by

$$\eta_j = C a_j e^{-i\omega t}$$

Taking only the real part,

$$\eta_j = \mathrm{Re}\left\{C a_j e^{-i\omega t}\right\} \qquad 8$$

Putting this equation (8) in equation (7),

We get,

$$T_{ij}\left\{C a_j (-i\omega)^2 e^{-i\omega t}\right\} + V_{ij}\left\{C a_j e^{-i\omega t}\right\} = 0$$

Or, $\quad -\omega^2 T_{ij} a_j + V_{ij} a_j = 0$

Or, $\quad \left(V_{ij} - \omega^2 T_{ij}\right) a_j = 0 \qquad 9$

Equation (9) constitutes n-linear homogenous equations for all a_j and consequently can have a nontrivial solution only if the determinant of the coefficients of all a_j vanishes.

That is, $\quad \left|V_{ij} - \omega^2 T_{ij}\right| = 0$

Expanding, we get,

$$\begin{vmatrix} V_{11} - \omega^2 T_{11} & V_{11} - \omega^2 T_{11} & \dots\dots\dots V_{1n} - \omega^2 T_{1n} \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & \dots\dots\dots V_{2n} - \omega^2 T_{2n} \\ . & & \\ . & & \\ . & & \\ V_{n1} - \omega^2 T_{n1} & V_{n2} - \omega^2 T_{n2} & \dots\dots\dots V_{nn} - \omega^2 T_{nn} \end{vmatrix} = 0 \qquad 10$$

Writing T_{ij} as an element of the matrix **T,** V_{ij} as an element of the matrix **V** and a_j as an element of eigenvector $\vec{a}$, then the equation (9) becomes,

$$(\mathbf{V}\text{-}\omega^2\mathbf{T})\ \vec{a}=0$$

Or, $(\mathbf{V}\text{-}\lambda\mathbf{T})\ \vec{a}=0$ where $\omega^2=\lambda$ (eigenvalue)

The characteristic equation of the n-roots is given by,

$$\left|\mathbf{V} - \lambda\mathbf{T}\right| = 0 \qquad 11$$

Where all λ are real and > 0.

The most general solution to equation (7) can be written as

$$\vec{\eta}(t) = \mathrm{Re}\left\{\sum_{k=1}^{N} C_k \vec{a}_k e^{-i\sqrt{\lambda_k}\ t}\right\} \qquad 12$$

With overlap of n-different frequencies which look like S. H. O. called fundamental modes of the system. All of the components of the matrix $\vec{a}_k$ should be real; otherwise in eigenvalue equation will not vanish.

To show $\vec{a}$ and λ are real, let $\vec{a}_k$ be a column matrix representing the kth eigenvector, satisfying the eigenvalue equation,

$$\mathbf{V}\,\vec{a}_k = \lambda_k \mathbf{T}\ \vec{a}_k \qquad 13$$

The adjoint equation for λ_l has the form

$$(\vec{a}_l)^\dagger\,\mathbf{V} = (\vec{a}_l)^{\dagger`}\,\mathbf{T}\ \lambda_l^* \qquad 14$$

Where $(\vec{a}_l)^\dagger$ denotes the adjoint vector-the complex conjugate row matrix and explicit use has been made of the fact that the **V** and **T** matrices are real and symmetric.

Multiplying 13 with $(\vec{a}_l)^\dagger$ from the left and 14 with $\vec{a}_k$ from the right yields

$$(\vec{a}_l)^\dagger\ \mathbf{V}\,\vec{a}_k = (\vec{a}_l)^\dagger\mathbf{T}\ \vec{a}_k\,\lambda_l^* \ = (\vec{a}_l)^\dagger\mathbf{T}\ \vec{a}_k\,\lambda_k \qquad 15$$

which means that for $k = l$ we must have λ_l real. Furthermore, we can choose the first (1) component for the $\vec{a}_k$ arbitrarily (homogenous equation 9) but then all other components are related to the first one through linear equations with all-real coefficients, so they will be real, too. Finally, 15 also shows that for different eigenvalues, the eigenvectors must be orthogonal in the sense that the product $(\vec{a}_l)^\dagger \mathbf{T}\ \vec{a}_k$ is zero. (This is all completely analog to quantum mechanics where the matrices T and V would be hermitian).

Example: Consider two masses m_1 and m_2 that are connected by three springs having spring constants k_1, k_2 and k_3 respectively. Then find Kinetic energy, potential energy, Lagrangian, Euler-Lagrange equation of motion and Eigenvalues of the system.

Solution:

Consider two masses m_1 and m_2 are connected by three springs having spring constants k_1, k_2 and k_3 respectively as shown in figure below:

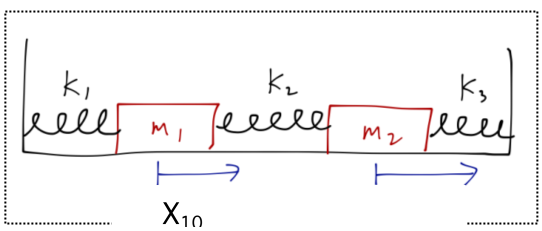

Figure: Small oscillation of two masses connected in three springs.

Let x_{10} and x_{20} are equilibrium coordinates of masses m_1 and m_2 respectively. So that the deviation from equilibrium position is given by relation,

$$\eta_i = x_i - x_{i0}$$

Where, $\eta_1 = x_1 - x_{10}$ and $\eta_2 = x_2 - x_{20}$

Which implies that $\dot{\eta}_1 = \dot{x}_1$ and $\dot{\eta}_2 = \dot{x}_2$

The kinetic energy of the system is given by

$$T=\frac{1}{2}m_1\dot{\eta}_1^2+\frac{1}{2}m_2\dot{\eta}_2^2 \qquad 1$$

Then the kinetic energy in the matrix form is

$$T=(\dot{\boldsymbol{\eta}})^T\mathbf{T}\dot{\boldsymbol{\eta}} \qquad 2$$

Where, $\boldsymbol{\eta}=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$ and $\mathbf{T}=\begin{pmatrix}m_1 & 0\\ 0 & \mathrm{m}_2\end{pmatrix}$

Again, the potential energy of the system is given by

$$V=\frac{1}{2}k_1\eta_1^2+\frac{1}{2}k_2(\eta_2-\eta_1)^2+\frac{1}{2}k_3\eta_2^2$$

Or, $$V=(\frac{k_1+k_2}{2})\eta_1^2-k_2\eta_1\eta_2+(\frac{k_2+k_3}{2})\eta_2^2 \qquad 3$$

Hence equations (1) and (3) are the required expressions for kinetic and potential energy of the system.

Then the potential energy in the matrix form is given by

$$V=(\boldsymbol{\eta})^T\mathbf{V}\boldsymbol{\eta} \qquad 4$$

Where, $\boldsymbol{\eta}=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$ and $\mathbf{V}=\begin{pmatrix}k_1+k_2 & -k_2\\ -k_2 & k_2+k_3\end{pmatrix}$

The Lagrangian is given by

$$L=T-V$$

$$=\left\{\frac{1}{2}m_1\dot{\eta}_1^2+\frac{1}{2}m_2\dot{\eta}_2^2\right\}-\left\{(\frac{k_1+k_2}{2})\eta_1^2-k_2\eta_1\eta_2+(\frac{k_2+k_3}{2})\eta_2^2\right\}$$

$$=\frac{1}{2}m_1\dot{\eta}_1^2+\frac{1}{2}m_2\dot{\eta}_2^2-(\frac{k_1+k_2}{2})\eta_1^2+k_2\eta_1\eta_2-(\frac{k_2+k_3}{2})\eta_2^2 \qquad 5$$

Writing the Euler-Lagrange equation with respect to η_1,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_1}\right) - \frac{\partial L}{\partial \eta_1} = 0$$

Or, $\frac{d}{dt}\left(\frac{1}{2} m_1\, 2\dot{\eta}_1\right) - \left(-\frac{k_1 + k_2}{2} 2\eta_1 + k_2\eta_2\right) = 0$

$\therefore \quad m_1\ddot{\eta}_1 = -(k_1 + k_2)\eta_1 - k_2\eta_2$ 6

Again, writing the Euler-Lagrange equation with respect to η_2,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_2}\right) - \frac{\partial L}{\partial \eta_2} = 0$$

Or, $\frac{d}{dt}\left(\frac{1}{2} m_2\, 2\dot{\eta}_2\right) - \left(k_2\eta_1 - \frac{k_2 + k_3}{2}\, 2\eta_2\right) = 0$

$\therefore \quad m_2\ddot{\eta}_2 = k_2\eta_1 - (k_2 + k_3)\eta_2$ 7

Hence equations (6) and (7) are the required Euler-Lagrange equations of motion of system.

We can write these equations (6) and (7) more compactly as a Matrix form of equation as bellow:

$$\begin{pmatrix} m_1 & 0 \\ 0 & \mathrm{m}_2 \end{pmatrix}\begin{pmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Or, more compactly,

$$\mathbf{T}\ddot{\boldsymbol{\eta}} = -\mathbf{V}\boldsymbol{\eta} \qquad 8$$

This is basically the usual equation for a simple harmonic oscillator in Matrix form. The simple structure of this equation motivates us to look for solutions in terms of Sine and Cosine.

Namely, let's try a solution where the entire vector $\boldsymbol{\eta}$ oscillates at a single frequency ω and the general solution is given by

$$\boldsymbol{\eta} = \mathrm{Re}\{C\,\mathbf{a_j}\,e^{-i\omega t}\}$$

Where, $\mathbf{a_j} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is a eigenvector.

Putting this solution in equation (8),

We get, $\mathbf{T}(-i\omega)^2\mathbf{a_j} = -\mathbf{V}\mathbf{a_j}$

Or, $$(\mathbf{V} - \omega^2\mathbf{T})\mathbf{a_j} = 0 \qquad 9$$

From this equation, the characteristic equation is

$$|\mathbf{V} - \omega^2\mathbf{T}| = 0$$

For convenience replacing $\omega^2$by λ, we get

$$|\mathbf{V} - \lambda\mathbf{T}| = 0$$

Or, $$\left|\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} - \lambda\begin{pmatrix} m_1 & 0 \\ 0 & \mathrm{m}_2 \end{pmatrix}\right| = 0$$

Or, $$\begin{vmatrix} k_1 + k_2 - \lambda m_1 & \text{-}\mathrm{k}_2 \\ -k_2 & k_1 + k_2 - \lambda m_1 \end{vmatrix} = 0$$

Or, $$(k_1 + k_2 - \lambda m_1)(k_2 + k_3 - \lambda m_2) - k_2^2 = 0$$

Or, $\lambda^2 m_1 m_2 - \lambda\{m_1(k_2 + k_3) + m_2(k_1 + k_2)\} + (k_1k_2 + k_1k_3 + k_2k_3) = 0$

This is quadratic equation in λ. To solve this equation, let us suppose: $m_1 = m_2 = m$ m$_1$=m$_2$=m and $k_3 = k_1$.

We get,

$$\lambda^2 m\ m\ - \lambda\{m\ (k_2 + k_1) + m\ (k_1 + k_2)\} + (k_1k_2 + k_1k_1 + k_2k_1) = 0$$

Or, $\lambda^2 m^2 - 2m\lambda(k_2 + k_1) + (k_1^2 + 2k_2k_1) = 0$

Or, $\lambda^2 - \frac{2}{m}(k_2 + k_1)\lambda + \left(\frac{k_1^2 + 2k_1k_2}{m^2}\right) = 0$

The solution of this quadratic equation is given by

$$\lambda = \frac{1}{2}\left\{\frac{2}{m}(k_1 + k_2) \pm \sqrt{\left\{-\frac{2}{m}(k_1 + k_2)\right\}^2 - 4\left(\frac{k_1^2 + 2k_1k_2}{m^2}\right)}\right\}$$

$$= \frac{k_1 + k_2}{m} \pm \frac{k_2}{m}$$

Taking positive sign: $\lambda = \frac{k_1 + 2k_2}{m}$

Taking negative sign: $\lambda = \frac{k_1}{m}$

Hence the required eigenvalues are $\frac{k_1 + 2k_2}{m}$ and $\frac{k_1}{m}$.

The corresponding eigenvectors are $\vec{a}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{a}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence, the eigenmodes of the system correspond to either a high frequency where the two masses move back on forth in opposite direction to each other, or a low frequency where they move in sync and the middle spring isn't stretched or compressed at all.