Special Relativity Lecture 1

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1 Relativity in the Context of Classical Mechanics

Here we start with the usual idea of two coordinate frames, S and S', with S' moving with constant velocity v with respect to S. Under this assumption, these frames are considered *inertial*. We will also limit ourselves to two spacetime dimensions, one spatial dimension (so we'll say along x) and one time dimension. We can come up with our first transformation rule regarding the spatial coordinate:

$$x' = x - vt \tag{1}$$

We can now predict the spatial coordinate of an object (or event) in the S' frame by measuring both time and space coordinates in S. The second transformation rule is even more simple, and would have been considered trivial before Einstein; the transformation for time:

$$t' = t \tag{2}$$

Observers in each frame, S' and S, will experience the same passage of time - they will always agree on the time intervals between events.

Since we are now beginning to deal with spacetime coordinates, it will prove convenient to portray our time dimension in the same units as our spatial dimension. We can do this by multiplying our time coordinate, t, by some value: $t \to ct$. This constant c can be any value that all observers can agree upon, and thus the constant value of the speed of light in a vacuum is traditionally chosen (in m/s if t has units of seconds)[1].

We can show this (linear) transformation in matrix form:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{3}$$

allowing us to transform one inertial coordinate system to another moving at constant velocity relative to the first.

[1] This is a useful trick in doing SR problems that deal with different units of time and/or distance. For example, you can change the units of c here to be $meters/year$ and the time unit to be $years$, etc., which usually results in solutions without numerical calculations

1.1 Invariance in Galilean Relativity

When we speak of "Galilean Relativity", we mean that the laws of nature are **invariant** under this type of transformation. Lets consider a (3D) Hamiltonian for a system of particles:

$$H = \sum_i \frac{\vec{p}_i^2}{2m_i} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j) \tag{4}$$

We are interested in how H changes in going from S to S' - which is actually a *canonical transformation*. How do we know? We can write the generator (which is of type 2):

$$F_2 = \sum_i \vec{r}_i \vec{P}_i + \vec{v} \cdot (\sum_i m_i \vec{r}_i - \sum_i \vec{P}_i t) \tag{5}$$

Where we've used a general notation. Does it work? We can see that we get our known transformations back:

$$\begin{aligned} \vec{P}_i &= \frac{\partial F_2}{\partial \vec{r}_i} = \vec{P}_i + \vec{v} m_i \\ \vec{R}_i &= \frac{\partial F_2}{\partial \vec{P}_i} = \vec{r}_i - \vec{v} t \end{aligned} \tag{6}$$

So yes, we have shown that this transformation in going from S to S' is canonical.

What happens to the actual Hamiltonian - what is the *kamiltonian*? By replacing the old variables with the new and not forgetting to add the explicit time dependence $\frac{\partial F_2}{\partial t} = -\sum_i \vec{v} \cdot \vec{P}_i$ (which ends up canceling out with the mixed term in the square), we find that:

$$K = \sum_i \frac{\vec{P}_i^2}{2m_i} + \sum_{i<j} V(\vec{R}_i - \vec{R}_j) + \sum_i \frac{m_i}{2} \vec{v}^2 \tag{7}$$

We can see that this has the same form as the old Hamiltonian, except we now have some constant term. However, we know that the Hamiltonian represents energy, and energy already has some form of offset. Thus, when we go to calculate the Hamiltonian equations of motion, this term is ignored and the resulting equations are the same except that they are represented in the new coordinates - the transformation leaves the equations of motion invariant.

An additional point one can make is that in the limit of very small $\vec{v}$, the canonical transformation above becomes an ICT, with the generator given by $\vec{G} = (\sum_i m_i \vec{r}_i - \sum_i \vec{P}_i t)$. (This is really a short-hand for 3 separate generators, each in one of the 3 cartesian directions). In this case, the term with the square velocity becomes higher order in an infinitesimally small parameter and can be ignored, but the linear term $-\sum_i \vec{v} \cdot \vec{P}_i$ does not get canceled, so that $\delta H = \{H, \vec{G}\} = -\sum_i \vec{P}_i$. It follows that $d\vec{G}/dt = \{\vec{G}, H\} + \partial \vec{G}/\partial t = \sum_i \vec{P} - \sum_i \vec{P} = 0$, i.e. $\vec{G}$ is conserved. This makes sense if we rewrite

$$\vec{G} = (\sum_i m_i \vec{r}_i - \sum_i \vec{P}_i t) = M\vec{R}_{CM} - M\dot{\vec{R}}_{CM} t \tag{8}$$

where M is the total mass of the system. Clearly, since $\vec{R}_{CM}(t) = \vec{R}_{CM}(t=0) + \dot{\vec{R}}_{CM} t$ in the absence of external forces, G just stands for the initial position of the center-of-mass.

2 Dynamical Equations from the Hamiltonian Approach

2.1 The Relativistic Hamiltonian

Unlike the usual way of starting with Einsteins postulate (that the speed of light is the same to all observers moving in inertial frames) in order to derive the dynamical equations of special relativity, we will begin with

Einsteins famous equation

$$m = \frac{E}{c^2} \tag{9}$$

which tells us that inertia depends on the total energy. We will use a primary feature of inertia; Newtons second law (for clarity, we will stick to 1 dimension).

$$F_x = \dot{p}_x = ma_x \tag{10}$$

Here, the Hamiltonian H is the total energy, so we can make the equivalence $H = E$. Our knowledge of Hamilton's equations allows us to make the following substitutions:

$$\begin{aligned} F_x = \dot{p}_x &= -\frac{\partial H}{\partial x} \\ \dot{x} &= \frac{\partial H}{\partial p_x} \end{aligned} \tag{11}$$

which results in the following form of Newtons second Law (with $m(H)$ substituted as well):

$$-\frac{\partial H}{\partial x} = \dot{p}_x = (\frac{H}{c^2})\frac{d}{dt}(\frac{\partial H}{\partial p_x}) \tag{12}$$

Since the total energy (H) is conserved and c is constant, we can move them into the total time derivative on the right hand side. Since we have a total time derivative on both sides, we have two perfect differentials and integration is trivial:

$$p_x = (\frac{H}{c^2})\frac{\partial H}{\partial p_x} + A \tag{13}$$

A (and later B) being the integration constant(s). This equation can now be integrated by separating variables:

$$\begin{aligned} \frac{1}{c^2}\int H dH &= \int (p_x + A) dp_x \\ \frac{H^2}{2c^2} &= \frac{p_x^2}{2} + Ap_x + B \\ H^2 &= c^2 p_x^2 + 2Ap_x c^2 + 2Bc^2 \end{aligned} \tag{14}$$

•In the case where $p_x = 0$, we must retrieve the rest energy (in terms of the rest mass m_0)

$$E = m_0 c^2 \tag{15}$$

We see that this must mean

$$\begin{aligned} 2Bc^2 &= m_0^2 c^4 \\ B &= \frac{m_0^2 c^2}{2} \end{aligned} \tag{16}$$

•In the case where p_x is small ($v << c$ where we are concerned), we must have classical correspondence, in that

$$E \approx m_0 c^2 + \frac{p_x^2}{2m_0} + ... \tag{17}$$

Squaring this, we get

$$E^2 = m_0^2 c^4 + p_x^2 c^2 + ... \tag{18}$$

This tells us that there should be no term linear in p_x in H^2; $A = 0$. We now have an expression for the Hamiltonian (or we can say the energy):

$$H^2 = E^2 = p_x^2 c^2 + m_0^2 c^4 \tag{19}$$

From here on note that we will drop the subscript x. From Hamilton's equations we know $\dot{x} = \partial H/\partial p$, so

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{pc^2}{\sqrt{p^2c^2 + m_0^2 c^4}} = \frac{pc^2}{E} \tag{20}$$

2.2 Energy, Momentum, and Velocity Transformations

Our primary goal now will be to write momentum and energy in one frame in terms of those in another frame moving relative to it with velocity v. We will also find our velocity addition formulas along with it. If the object in question is positioned at the origin of the S' frame (stationary in S'), then we know the energy of that object measured in this frame is its rest energy, $E_{S'} = E_0 = m_0c^2$.

In S, S' (and thus the object), is moving at velocity v. From the calculation above (19), we can write an expression for its momentum in terms of this velocity:

$$p = \frac{v}{c}\frac{E_S}{c} = \beta\frac{E_S}{c} \tag{21}$$

We can then substitute this into (17):

$$\begin{aligned} E_S^2 - p^2c^2 &= m_0^2c^4 \\ E_S^2(1-\beta^2) &= m_0^2c^4 \\ E_S &= \gamma m_0c^2 \end{aligned} \tag{22}$$

or, $E_S = \gamma E_{S'}$. Plugging this value for E_S back into P, we can get the momentum in the S frame in terms of the rest energy $E_{S'}$.

$$p_S = \gamma\beta\frac{E_{S'}}{c} \tag{23}$$

We are looking for transformation equations that behave the same way going back and fourth between coordinate frames. Thus, our first guess at the inverse equation $p_{S'}$ would be the same as (22), but with the velocity negated (due to frame S having apparent motion in the negative x direction in S'). However, this would be incorrect; we are missing another term. To make it consistent, we can write:

$$\begin{aligned} p_{S'} &= -\frac{\gamma\beta E_S}{c} + \gamma p_S \\ &= -\gamma^2\beta\frac{E_{S'}}{c} + \gamma^2\beta\frac{E_{S'}}{c} \\ &= 0 \end{aligned} \tag{24}$$

Similarly for the Energy,

$$\begin{aligned} E_{S'} &= \gamma E_S - \gamma c\beta p_S \\ &= \gamma^2 E_{S'} - \gamma^2\beta^2 E_{S'} \\ &= E_{S'} \end{aligned} \tag{25}$$

These equations, (23) and (24), must be the most general set of transformations to go from the S to S' frames.

We can now calculate $v_{S'}$ in S' if we know v_S in S (now lifting the assumption that the object was stationary in S'). From (19), we will substitute in (23) and (24):

$$\begin{aligned} v_{S'} &= \frac{p_{S'}}{E_{S'}}c^2 \\ &= \frac{(\gamma p_S - \gamma\beta\frac{E_S}{c})}{(\gamma E_S - \gamma c\beta p_S)}c^2 \end{aligned} \tag{26}$$

If we divide the numerator and denominator by E_S, and noting that $v_S = \frac{p_Sc^2}{E_S}$, we can find:

$$v_{S'} = \frac{v_S - v}{1 - \frac{v_s v}{c^2}} \tag{27}$$

This is our well-known velocity transformation equation.

2.3 The Lorentz Transformation

Noting that $v_S = \frac{dx}{dt}$ and $v_{S'} = \frac{dx'}{dt'}$, we can write (26) in the following form:

$$\frac{1}{c}\frac{dx'}{dt'} = \frac{dx - vdt}{cdt - \beta dx} \tag{28}$$

which gives us the form of the transformation equations for position and time between S and S'. This transformation is the **Lorentz transformation**, and in one dimension it can be written in matrix form as:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \bar{L} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{29}$$

Note that this boosts us from the S frame to the S' frame. If we then want to go BACK to the S frame, we would simply negate the velocities in $\bar{L}$ (for the same reasons outlined in the previous section). This gives us:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} = \bar{L}^{-1} \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{30}$$

I have jumped a step in assuming that this matrix is the inverse of $\bar{L}$ in (28), but you can prove that it is the inverse by taking the matrix product:

$$\bar{L}\bar{L}^{-1} = \bar{1} \tag{31}$$

This is expected, if we were to apply these transformations consecutively, we would hope that we should get the original components back - which we clearly do.

The Lorentz transformation can be generalized for any velocity vector in three spatial dimensions:

$$\bar{L} = \begin{pmatrix} \gamma & -\gamma\vec{\beta} \\ -\gamma\vec{\beta} & \bar{A} \end{pmatrix} \tag{32}$$

Where $\bar{A}$ is the matrix defined by

$$\bar{A} = \bar{1} + (\gamma - 1)\frac{\vec{\beta} o \vec{\beta}}{\beta^2} \tag{33}$$