Galilean Relativity

$$S \to S' \quad V_x \quad x' = x - \frac{V}{c}ct \quad ct' = ct \quad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{V}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j)$$

$$F_2 = \sum_i \vec{r}_i \cdot \vec{P}_i + \vec{V}\left(\sum m_i \vec{r}_i - \sum_i \vec{P}_i t\right)$$

$$\vec{p}_i = \frac{\partial F_2}{\partial \vec{r}_i} = \vec{P}_i + \vec{V} m_i \Rightarrow \vec{P}_i = \vec{p}_i - m_i \vec{V}$$

$$\vec{R}_i = \frac{\partial F_2}{\partial \vec{P}} = \vec{r}_i - \vec{v} t \qquad -\frac{\partial F_2}{\partial t} = \vec{V} \sum \vec{P}_i$$

$$K = \sum_i \frac{(\vec{P}_i - m_i \vec{v})^2}{2m_i} + \sum_{i<j} V(\vec{R}_i - \vec{R}_j)$$

$$= \sum_i \frac{\vec{P}_i^{\,2}}{2m_i} + \sum_{i<j} V(\vec{R}_i - \vec{R}_j) + \sum_i \frac{m_i}{2} \vec{V}^2$$

$$\text{ICT:} \quad F_2 = \sum \vec{r}_i \vec{P}_i + \delta \vec{v}\, \vec{G} \qquad \vec{G} = \sum m_i \vec{r}_i - \sum \vec{P}_i t$$

GENERATOR CONSERVED for ICT (note: in that case, $\frac{\partial F_2}{\partial t}$ is not subtracted):
$\delta H = \{H, \vec{G}\}\delta\vec{v} = -\delta\vec{v}\sum_i \vec{p}_i + O(\delta\vec{v}^2)$ (c.f. to the Kamiltonian above).
So $\frac{dG}{dt} = \{\vec{G}, H\} + \frac{\partial \vec{G}}{\partial t} = +\sum_i \vec{p}_i - \sum_i \vec{p}_i = 0$, i.e. the Generator is a conserved quantity.
We can rewrite it as $\vec{G} = M\vec{R} - \vec{P}t$ where $\vec{R}$ is the center-of-mass, *M* is the total mass, and $\vec{P} = M\vec{V}$ is the total momentum of the system. Of course, it is obvious why the generator is conserved (in the absence of external forces) – it is identical to *M* times the center of mass at *t* = 0 (the center of mass moves according to $\vec{R}(t) = \vec{R}_0 + \vec{V}t$.

Deriving the equations of special relativity from Einstein’s most famous Equation:

$$E = mc^2 \rightarrow m = \frac{E}{c^2}, \quad \vec{F} = m\vec{a} \rightarrow -\frac{\partial H}{\partial x} = \frac{H}{c^2}\frac{d}{dt}\frac{\partial H}{\partial p} \rightarrow$$

$$\dot{p} = \frac{d}{dt}\frac{H}{c^2}\frac{\partial H}{\partial p} \rightarrow \frac{1}{c^2}\frac{1}{2}\frac{\partial H^2}{\partial p} = p + A \rightarrow \frac{1}{c^2}\frac{1}{2}H^2 = \frac{1}{2}p^2 + Ap + B$$

$$H^2 = p^2c^2 + \cancel{2Apc^2} + \underbrace{2Bc^2}_{m_0^2c^4} \qquad 1)\ p = 0 \rightarrow E_0 = m_0c^2 \text{ rest mass}$$

$$2)\ p \text{ small} \rightarrow \left(E \simeq m_0c^2 + \frac{p^2}{2m_0} + \cdots\right)^2$$

$$E^2 = m_0^2c^4 + p^2c^2 + \cdots$$

$$H^2 = E^2 = p^2c^2 + m_0^2c^4$$

$$H = \sqrt{p^2c^2 + m_0^2c^4} \qquad \dot{x} = \frac{\partial H}{\partial p} = \frac{1}{2}\frac{1}{E}2pc^2 = \frac{pc}{E}c$$

$$S' : E = E_0 = m_0c^2$$

$$S : p = \frac{v}{c}\frac{E}{c} \qquad E^2 - p^2c^2 = m_0^2c^4$$

$$E^2\left(1 - \frac{v^2}{c^2}\right) \Rightarrow E_S = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} = \gamma \underbrace{mc^2}_{E_{S'}}$$

$$p_S = \gamma \overset{\shortparallel}{\beta} \underbrace{m_0c^2}_{E_{S'}}$$

$$p_{S'} = -\gamma\beta E_S + \gamma p_S = 0 \qquad E_{S'} = \gamma E_S - \gamma\beta p_S$$

$$v_{S'} = p_{S'}\frac{c^2}{E_{S'}} = \frac{\gamma p_S - \gamma\beta E_S}{\gamma E_S - \gamma\beta p_S}c^2 = \frac{v_S - v}{1 - v_S v/c^2} = dx'/dt' = \frac{dx - v\,dt}{dt - dx\,v/c^2}$$

The last line is the well-known rule for velocity addition. The last equality shows how *dx’* and *dt’* must translate from *dx* and *dt* (together with the requirement that the transformation is described by a matrix which becomes its own inverse simply by changing the sign of *v*) => Lorentz Transformation:

$$\Rightarrow \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \qquad \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \text{LT}$$

$$L = \begin{pmatrix} \gamma & -\vec{\beta} \\ -\vec{\beta} & \boxed{} = \mathbb{1} + (\gamma-1)\dfrac{\vec{\beta}\circ\vec{\beta}}{\beta^2} \end{pmatrix}$$

$$\begin{pmatrix} ct' \\ \vec{r}\,' \end{pmatrix} = \begin{pmatrix} L \end{pmatrix} \begin{pmatrix} ct \\ \vec{r} \end{pmatrix}$$

$(\Delta s^2) = \Delta c^2 t^2 - \Delta \vec{r}^{\,2}$ Lorentz invariant

$\Delta s^2 > 0 \rightarrow$ time like distance $\Delta\tau = \frac{\sqrt{\Delta s^2}}{c}$

$\Delta s^2 = 0 \rightarrow$ light-like distance

$\Delta s^2 \leq 0 \rightarrow$ space-like separation $(ct, \vec{r})$

3D $\rightarrow$ 4D 4-vectors (4-forms, 4 tensors ...) $(E, \vec{p}c)$