Conservation Laws

q_i, $\dot q_i$, $\mathcal{L}(q_i, \dot q_i, t)$

IF $\mathcal{L}$ doesn't depend on q_k $\rightarrow$ cyclic, ignorable $\rightarrow$ $p_k = \text{const}$

Emmy Noether Theorem:

transformation that leaves laws of nature invariant $\longleftrightarrow$ conserved quantity

Def.: $\frac{\partial \mathcal{L}}{\partial \dot q_i} = p_i$ generalized momentum $\Rightarrow \frac{d}{dt} p_i = \frac{\partial \mathcal{L}}{\partial q_i}$ $(\dot p = F)$

Ex.1: $\mathcal{L} = \frac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2) - V(x, y, \cancel{z})$

$\frac{\partial \mathcal{L}}{\partial z} = 0 \Rightarrow p_z$ conserved

$p_x = \frac{\partial \mathcal{L}}{\partial \dot x} = m\dot x$

y: $m\dot y$

z: $m\dot z$

Ex.2: $\mathcal{L} = \frac{m}{2}(\dot r^2 + r^2 \dot\varphi^2) - V(r)$

$\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \Rightarrow p_\varphi$ ~ conserved

$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot\varphi} = mr^2\dot\varphi = L_z$ = const

$p_r = \frac{\partial \mathcal{L}}{\partial \dot r} = m\dot r$ not conserved

$\dot\varphi = \frac{p_\varphi}{mr^2}$

$m\ddot r = mr\dot\varphi^2 - \frac{\partial V}{\partial r} = \frac{p_\varphi^2}{mr^3} - \frac{\partial V}{\partial r}$

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h-function

$h(q_i, \dot q_i, t) = \sum_i \underbrace{p_i}_{=\frac{\partial \mathcal{L}}{\partial \dot q_i}} \dot q_i - \mathcal{L}(q_i, \dot q_i, t)$

$h = E$? $\Longleftrightarrow$ $2T$ $\Rightarrow$ $h = T + V$

Ex.1: $h = \underbrace{m\dot x^2 + m\dot y^2 + m\dot z^2 - \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2)}_{T} + V(x, y, z)$ ✓

Counter-Ex.:
HW Set 1, Problem 2

$T = \frac{m}{2}\left[(\dot q_1 - \omega q_2)^2 + (\dot q_2 + \omega q_1)^2\right]$

$p_1 = m(\dot q_1 - \omega q_2)$

$p_2 = m(\dot q_2 + \omega q_1)$

$h = p_1 \dot q_1 + p_2 \dot q_2 = m\left[\dot q_1^2 - \omega q_2 \dot q_1 + m\dot q_2^2 + \omega q_1 \dot q_2\right] - \ldots \neq T$ ✗

Conserved?

$\frac{d}{dt} h = \sum_i \left[\dot p_i \dot q_i + p_i \ddot q_i - \underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{\dot p_i} \dot q_i - \frac{\partial \mathcal{L}}{\partial \dot q_i} \ddot q_i\right] - \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$

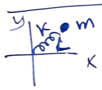
IF yes $h = \text{const.} = $ "E"

m_1, m_2 ; cartesian coordinates : $\mathcal{L} = \frac{M}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{\mu}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x,y,z)$

$$\underbrace{P_x = M\dot{X},\ P_y = M\dot{Y},\ P_z = M\dot{Z}}_{\vec{P}\ =\ \text{conserved}}$$

$\rightarrow X(t) = X_0 + \frac{P_x}{M}t,\ Y(t),\ Z(t) \ldots$; $h = T + V$ Energy ✓ conserved ✓

"E" = const. + $T(\dot{\vec{r}}) + V(\vec{r})$



$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{k}{2}(r-L)^2$$

φ ignorable $\rightarrow p_\varphi = mr^2\dot{\varphi} =$ const.

$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$

"E" $h = p_r\dot{r} + p_\varphi\dot{\varphi} - \mathcal{L} = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\varphi}^2 + \frac{k}{2}(r-L)^2$ conserved ✓

$$E = \underbrace{\frac{m}{2}\dot{r}^2}_{\text{"}T_{kin}\text{"}} + \underbrace{\frac{p_\varphi^2}{2mr^2} + \frac{k}{2}(r-L)^2}_{\text{"}V\text{"}(r)} \Rightarrow m\ddot{r} = -\frac{\partial \text{"}V\text{"}}{\partial r} = \frac{p_\varphi^2}{mr^3} - k(r-L)$$

$\Rsh\ r_0(\varphi_0, p_\varphi)$

Stationary: $\frac{\partial \text{"}V\text{"}}{\partial r} = 0 \longrightarrow \frac{p_\varphi^2}{mr^3} \overset{!}{=} k(r-L)$

$$\boxed{\text{"}V\text{"}(r) = \text{"}V\text{"}(r_0) + \cancel{\frac{\partial \text{"}V\text{"}}{\partial r}(r-r_0)} + \frac{1}{2}\frac{\partial^2 \text{"}V\text{"}}{\partial r^2}(r-r_0)^2}$$

Small excursion around r_0