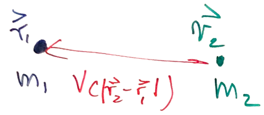
2-body problem with $V(r)$



$$\mathcal{L} = \cancel{\frac{M}{2}(\underbrace{\dot X^2+\dot Y^2+\dot Z^2}_{\dot{\vec R}^2})} + \frac{\mu}{2}(\underbrace{\dot x^2+\dot y^2+\dot z^2}_{\dot r^2}) - V(r)$$

$$\vec P = M\dot{\vec R} = \text{const} = 0 \text{ by definition of coordinate system}$$

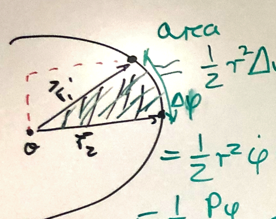
$$\Rightarrow \mathcal{L} = \frac{\mu}{2}(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\varphi^2) - V(r) = \frac{\mu}{2}(\dot r^2 + r^2\dot\varphi^2) - V(r)$$

choose $\hat x, \hat y, \hat z$ such that $\theta = 90°$ @ $t=0$, $\dot\theta = 0$, $\ddot\theta = 0$

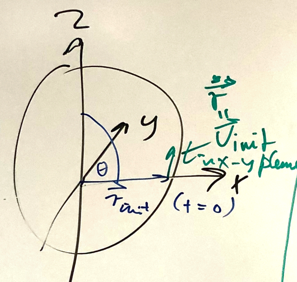
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{d}{dt}(\mu r^2\dot\theta) = \cancel{2\mu r\dot r\underbrace{\dot\theta}_{0}} + \underbrace{\mu r^2\ddot\theta}_{\text{must}=0} = \frac{\partial\mathcal{L}}{\partial\theta} = \cancel{\mu r\sin\theta\underbrace{\cos\theta}_{0}\dot\varphi^2}$$

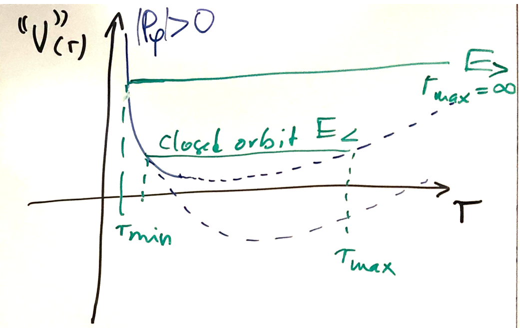
at $t=0$:

$$p_\varphi = \frac{\partial\mathcal{L}}{\partial\dot\varphi} = \mu r^2\dot\varphi = \text{const} = \mu|\vec r\times\vec v| = L_z \Rightarrow \text{Kepler's 2nd law} ✓$$



$$= \frac{1}{2}r^2\dot\varphi\,\Delta t = \frac{1}{2}\frac{p_\varphi}{\mu}\Delta t$$





$$p_r = \frac{\partial\mathcal{L}}{\partial\dot r} = \mu\dot r \Rightarrow h = \cancel{\mu\dot r^2} + \cancel{\mu r^2\dot\varphi^2} + \frac{\mu}{2}(\dot r^2 + r^2\dot\varphi^2) + V(r) = E = \text{cons.} \qquad \frac{dh}{dt} = \frac{\partial h}{\partial t}$$

$$= \frac{\mu}{2}\dot r^2 + \underbrace{\frac{p_\varphi^2}{2\mu r^2} + V(r)}_{\text{"V"}(r)} \rightarrow \mu\ddot r = -\frac{\partial\text{"V"}}{\partial r}$$

$$\frac{\mu}{2}\dot r^2 = E - \frac{p_\varphi^2}{2\mu r^2} - V(r) \qquad \frac{dr}{dt} = \dot r = \sqrt{\underbrace{\frac{2E}{\mu} - \frac{p_\varphi^2}{\mu^2 r^2} - \frac{2V(r)}{\mu}}_{\frac{2E}{\mu} - \frac{2\text{"V"}}{\mu}}} \Rightarrow \underbrace{\int_{r_0 = r_{min}}^{r_f \le r_{max}} \frac{dr}{\sqrt{\quad}}}_{F(r_f)} = \int_{t=0}^{t_f \le t_{max}} dt = t_f$$

$$r(t) = F^{-1}(t) \leftarrow F(r_f)$$

$$T_{orbit} = 2t_{max}$$

$L = 0$ = relaxed

m $\overset{k}{\sim\sim\sim}$ m

$\mu = \frac{m}{2}$

$$\text{"}V\text{"}(r) = \frac{p_\varphi^2}{2\mu r^2} + \frac{k}{2} r^2$$

Equilibrium? $\frac{\partial \text{"}V\text{"}}{\partial r} = 0 \Rightarrow -2\frac{p_\varphi^2}{2\mu r^3} + kr = 0$

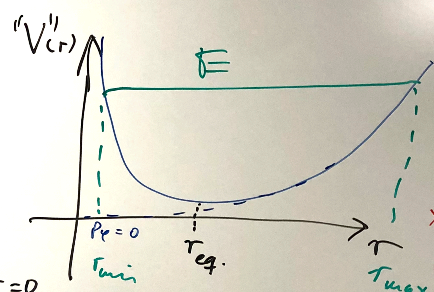
$$r_{eq}^4 = \frac{p_\varphi^2}{\mu k}$$

Small osc.:

$$\text{"}V\text{"} = V(r_{eq}) + \underbrace{\frac{1}{2}\frac{\partial^2 \text{"}V\text{"}}{\partial r^2}}_{\underbrace{\frac{1}{2}\left(3\frac{p_\varphi^2}{\mu r_{eq}^4} + k\right)}_{\text{"}k\text{"}}} (r - r_{eq})^2$$

$$\Rightarrow \omega \simeq \sqrt{\frac{\text{"}k\text{"}}{\mu}} \Downarrow \sqrt{\frac{4k}{\mu}}$$

$$T_{osc} = \frac{2\pi}{\omega} \simeq 2\pi\sqrt{\frac{\mu}{k}}$$



$\Rightarrow$ Orbit area $\pi r_{eq}^2 \Rightarrow T_{orbit} = \frac{2\pi\mu r_{eq}^2}{p_\varphi}$

$$\omega = \sqrt{\frac{k}{\mu}}$$

$$T = \frac{\mu}{2}(\dot{x}^2 + \dot{y}^2)$$

$$V = \frac{k}{2}(x^2 + y^2)$$

$$x = A_x \cos(\omega t + \varphi_x)$$

$$y = A_y \cos(\omega t + \varphi_y)$$

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r_{min}^2: "V"(r) = E

$$E r_{min}^2 - \frac{p_\varphi^2}{2\mu} - \frac{k}{2} r_{min}^4 = 0 \rightarrow r^2_{\substack{max \\ min}} = \frac{E}{k} \pm \sqrt{\left(\frac{E}{k}\right)^2 - \frac{p_\varphi^2}{\mu k}}$$

$$T_{orbit} = 2\,T_{max} = 2\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{\frac{2E}{\mu} - \frac{2\text{"}V\text{"}(r)}{\mu}}}$$