$H(\vec{q}, \vec{p}, t)$ sometimes $= E$ and sometimes conserved.

1) $\frac{\partial H}{\partial q_i} = -\dot{p}_i$ 2) $\frac{\partial H}{\partial p_i} = \dot{q}_i$ HEM

From variation principle

$$\int_{t_1}^{t_2} \sum_i p_i \dot{q}_i - H(\vec{q}, \vec{p}, t) dt$$ must be stationary $(\delta S = 0)$

along $\forall$ path in phase space with

$\vec{q}(t_1), \vec{p}(t_1), \vec{q}(t_2), \vec{p}(t_2)$ fixed

$\left. \begin{matrix} q_1 \dots q_K \\ p_1 \dots p_K \end{matrix} \right\} 2K$

We can describe system in phase space

1) shows that if there is no q_i in H $\Rightarrow$ $p_i = \text{const}$ $\Rightarrow$ we can integrate the expression. If H doesn't depend on all q_i-s $\Rightarrow$ the problem becomes trivial. It can happen when problem is trivial or if we choose the generalised coord-s well.

We must be able to change variables from one to another set. We must be able replace

$$\boxed{\vec{q}, \vec{p} \rightarrow \vec{Q}, \vec{P}} \quad (\vec{q}, \vec{p}, t) \Rightarrow H \rightarrow K(\vec{Q}, \vec{P}, t)$$

cannonical trans-n (point transformation in phase space)

K can be $=$ to H. Require $\frac{\partial K}{\partial Q_i} = -\dot{P}_i$, $\frac{\partial K}{\partial P_i} = \dot{Q}_i$

If $Q_i = Q_i(\vec{q}, t)$ only $\rightarrow$ point transformation in configuration space

transf-n is cannonical means that also

$$\delta \int_{t_1}^{t_2} \sum_i P_i \dot{Q}_i - K(\vec{Q}, \vec{P}, t) dt = 0$$ (it also has to be stationary)

If we can add something to $\int$ from which the $\delta\int$ automatically becomes 0 => $\delta\int$ is still 0

$$\delta\int_{t_1}^{t_2}\left\{\sum P_i\dot{Q}_i - K(\vec{Q},\vec{P},t) + \frac{dF}{dt}\right\}dt = 0.$$

$$\int = F(t_2) - F(t_1)$$

F is generating function

F is an arbitrary f-n which is const at t_2 and t_1 (beginning + end of path)

$\delta\int$ will not also change if we * it with λ.

We call it <u>scale transformation</u>

$$\delta\int\lambda\left(\sum p_i\dot{q}_i - H(\vec{q},\vec{p},t)\right)dt = 0.$$

<u>Example 1</u> $P_i = \mu p_i$, $Q_i = \nu q_i$, $K = \underbrace{\mu\nu}_{\lambda} H$

$$\frac{\partial K}{\partial Q_i} = \frac{1}{\nu}\nu\mu\frac{\partial H}{\partial q_i} = -\mu\dot{p}_i = -\dot{P}_i$$

and the same for the second eq.-n.:

$$\frac{\partial K}{\partial P_i} = \frac{1}{\mu}\frac{\partial K}{\partial p_i} = \frac{1}{\mu}\mu\nu\frac{\partial H}{\partial p_i} = \nu\dot{q}_i = \dot{Q}_i$$

$$\int\sum_i P_i\dot{Q}_i - K(\vec{Q},\vec{P},t)dt = \int\sum\mu p_i\nu\dot{q}_i - \mu\nu H(\vec{q},\vec{p},t)\,dt$$
$$= \int\lambda\left[\sum p_i\dot{q}_i - H\right]dt$$

Assume we've chosen our coordinates such that $\lambda = 1$.
For the new variables we should have following cond.-n

$$\boxed{\sum p_i\dot{q}_i - H(\vec{q},\vec{p},t) = \sum P_i\dot{Q}_i - K(\vec{Q},\vec{P},t) + \frac{dF}{dt}} \quad (*)$$

If we can prove this, then we'll have cannonical transform-n

Example Type 1 gen. f-n $F_1(\vec{q}, \vec{Q}, t)$

Example: $F_1 = \sum_i q_i Q_i$ $\quad \frac{dF}{dt} = \sum_i \dot{q}_i Q_i + q_i \dot{Q}_i$

In $(*)$ neither K nor H depend on $\dot{q}_i$ or $\dot{Q}_i$

$$\sum_i p_i \dot{q}_i - \sum_i \dot{q}_i Q_i = \sum_i P_i \dot{Q}_i + H(\vec{q}, \vec{p}, t) - K(\vec{Q}, \vec{P}, t) + \sum_i q_i \dot{Q}_i$$

$\sum (p_i - Q_i)\dot{q}_i$

nothing in right side depends on $\dot{q}_i$

$\Rightarrow \quad p_i - Q_i = 0$

$\Rightarrow \boxed{Q_i = p_i}$

if we do same for $\dot{Q}_i$ $\Rightarrow \sum_i (P_i + q_i)\dot{Q}_i \Rightarrow \boxed{P_i = -q_i}$

$$\Rightarrow \underline{H = K}$$

Example 3

Freely falling marble in 1-dimension.

(q - height)

$$H = \frac{p^2}{2m} + mgq$$

$Q = p \quad , \quad P = -q$

$$\Rightarrow K = \frac{Q^2}{2m} - mgP$$

$$\frac{\partial K}{\partial P} = \dot{Q} = -mg \qquad \qquad \frac{\partial K}{\partial Q} = -\dot{P} = \frac{Q}{m}$$

$$\dot{p} = -mg \qquad \qquad \dot{q} = \frac{p}{m}$$

Example 4 general case.

$$F_1(\vec{q}, \vec{Q}, t) \quad , \quad \frac{dF_1}{dt} = \sum_i \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial Q_i} \dot{Q}_i \right) + \frac{\partial F}{\partial t}$$

⇒ Again doing same steps as in example 2), from (*) we get;

$$\sum \left(p_i - \frac{\partial F_1}{\partial q_i}\right) \dot{q}_i \Rightarrow \boxed{p_i = \frac{\partial F_1}{\partial q_i}} = f_i(\vec{q}, \vec{Q}, t) \rightarrow \text{solve for } \vec{Q}(\vec{q}, \vec{p}, t)$$

and

$$\sum \left(P_i + \frac{\partial F_1}{\partial Q_i}\right) \dot{Q}_i \Rightarrow \boxed{P_i = -\frac{\partial F_1}{\partial Q_i}} = g_i(\vec{Q}, \vec{q}, t) \rightarrow P_i(\vec{q}, \vec{p}, t)$$

$$\Rightarrow K = H + \frac{\partial F}{\partial t}$$, the transformation is cannonical

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These steps we do if we're given F.

But if we know Q, P, we need to find F?

⇒ we have $\frac{\partial F_1}{\partial q_i} = p_i$ and $P_i = -\frac{\partial F_1}{\partial Q_i}$ which we need to integrate to find F_1

these we can integrate if they fulfill condition

$$\frac{\partial^2 F_1}{\partial q_i \partial Q_i} = \frac{\partial^2 F_1}{\partial Q_i \partial q_i} \quad \text{or} \quad \frac{\partial}{\partial q_i}(-P_j) = \frac{\partial}{\partial Q_i}(p_i)$$

Example 5 Harmonic oscillator: q, $p = m\dot{q}$

$$p = m\omega A \cos\omega t$$
$$q = A \sin\omega t$$

$$H = \frac{p^2}{2m} + \frac{1}{2} k q^2 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \qquad \omega^2 = \frac{k}{m}$$

We want to choose Q such that $H = H(Q)$ and P = const