3/30/06 Poisson Brackets continued

$$[\quad,\quad] \longrightarrow [u,v] = -[v,u]$$

$u(\vec{\eta},t)$ $v(\vec{\eta},t)$

$$[au_1 + bu_2, v] = a[u_1, v] + b[u_2, v]$$

$$[u_1 \cdot u_2, v] = u_1[u_2, v] + u_2[u_1, v]$$

Jacobi Identity

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

Proof left as an exercise for the reader

$$[u,v] = \left(\frac{\partial u}{\partial \vec{\eta}}\right)^T \bar{J} \left(\frac{\partial v}{\partial \vec{\eta}}\right), \text{ so}$$

$$[u,[v,w]] = \left(\frac{\partial u}{\partial \vec{\eta}}\right)^T \bar{J} \left(\frac{\partial}{\partial \vec{\eta}}\left(\left(\frac{\partial v}{\partial \vec{\eta}}\right)^T \bar{J} \left(\frac{\partial w}{\partial \vec{\eta}}\right)\right)\right)$$

in Q.M. $[u,v] \rightarrow \frac{1}{i\hbar}[U, V]$

Reformulating Classical Mechanics in the Poisson Bracket Formalism

Recall: ICT: $\vec{\eta} + d\vec{\eta} = \vec{\xi}$;

$$F_2 = \sum q_i P_i + \varepsilon G(\vec{q}, \vec{P}, t), \text{ so}$$

$$\delta\vec{\eta} = \varepsilon \bar{J}\left(\frac{\partial G}{\partial \vec{\eta}}\right)$$

which implies

$$\delta\vec{\eta} = \varepsilon[\vec{\eta}, G],$$

since $[\eta_i, G] = \left(\frac{\partial \eta_i}{\partial \vec{\eta}}\right) \bar{J} \left(\frac{\partial G}{\partial \vec{\eta}}\right)$

and
$$\left(\frac{\partial \vec{\eta}}{\partial \eta_\gamma}\right) = \begin{array}{c}\left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \end{array}\right]\end{array}\begin{array}{l} \\ \\ \\ \gamma \\ \\ \\ \\ \end{array}$$

Example : $G \equiv H$

$$\delta\vec{\eta}(dt) = dt\,\bar{J}\left(\frac{\partial H}{\partial \vec{\eta}}\right) = dt\,[\vec{\eta}, H]$$

$$u(\varepsilon) \xrightarrow{\varepsilon} u + \delta u \;;\quad \delta u = \left(\frac{\partial u}{\partial \vec{\eta}}\right)^T \delta\vec{\eta}$$

$$= \varepsilon \left(\frac{\partial u}{\partial \vec{\eta}}\right)^T \bar{J} \left(\frac{\partial G}{\partial \vec{\eta}}\right)$$

$$= \varepsilon[u, G]$$

If u is not explicitly dependent on time,

$$\dot{u} = [u, H] + \frac{\partial u}{\partial t} = [u, H]$$

So, if we are interested in small variations in Energy (and H has no explicit time dependence), then

$$\delta H = \varepsilon[H, G]$$

Emmy Noether: If a change in ε, G leaves system invariant ($\delta H = 0$), then G is conserved

(→)

Recall $q_j \to q_j + \delta q$, then $G \equiv P_j \Rightarrow$ if H is invariant under translation in q_j-direction $\Rightarrow P_j$ is conserved

Once we discover two conserved quantities, we can find more ...

Suppose G_1, G_2 are conserved ~~and~~ $\frac{\partial G_1}{\partial t} = \frac{\partial G_2}{\partial t} = 0$, then

$$[G_1, H] = 0 \; ; \; [G_2, H] = 0 ,$$

and

$$[H, [G_1, G_2]] = -[G_1, \underbrace{[G_2, H]}_{0}] - [G_2 \underbrace{[H, G_1]}_{0}]$$

which implies $[G_1, G_2]$ is invariant. (see HW PS #9)

If we know $u(t)$, where $u(0) = u_0$ for any system, where u contains combinations of position & momentum, then the system is essentially solved. more generally, let $\alpha \in \mathbb{R}$. Then for a specific transformation given by G and α

$$\eta(\alpha) \to u(\alpha) \; ; \; u(\alpha = 0) = u_0$$

and

$$\delta\vec{\eta} = d\alpha \, \mathbb{J}\left(\frac{\partial G}{\partial \vec{\eta}}\right) \; ; \text{ where } G = G(\vec{\eta})$$

So, expand u near α,

$$u(\alpha) = u_0 + \frac{du}{d\alpha}\bigg|_{\alpha=0} \alpha + \frac{1}{2}\frac{d^2u}{d\alpha^2}\bigg|_{\alpha=0} \alpha^2 + \ldots$$

$$= u_0 + [u, G]_{\alpha=0}\,\alpha + \frac{1}{2}[[u, G], G]\,\alpha^2 + \ldots$$

Example: Marble in free fall

$$H = \frac{p^2}{2m} + mgy \; ;$$

$$y(t) = y_0 + \underbrace{[y, H]}_{\frac{\partial H}{\partial p} = \frac{p}{m}} t - \frac{1}{2} g \underbrace{\left[\frac{p}{m}, H\right]}_{-\frac{1}{2m}\left(\frac{\partial H}{\partial y}\right) = -\frac{g}{2}} t^2 \Rightarrow$$

higher brackets are zero

$$y(t) = y_0 + \frac{p}{m} t - \frac{1}{2} g t^2 \quad \text{q.e.d.}$$

Example: What is the rotation generator?
Suppose we have 6 D.O.F $\Rightarrow$ 6 generalized coordinates (x, y, z, p_x, p_y, p_z), and we rotate by $d\theta$:

$$x \to x - y\,d\theta$$
$$y \to y + x\,d\theta$$

So,

$$\delta\vec{\eta} = d\theta\left(\bar{J}\right)\left(\frac{\partial G}{\partial\vec{\eta}}\right)$$

and $$dx = d\theta\left(\frac{\partial G}{\partial p_x}\right); \quad dy = d\theta\left(\frac{\partial G}{\partial p_y}\right)$$

$$\partial p_x = -d\theta\left(\frac{\partial G}{\partial x}\right); \quad \partial p_y = -d\theta\left(\frac{\partial G}{\partial y}\right)$$

Therefore

$$-y = \frac{\partial G}{\partial p_x}; \quad x = \frac{\partial G}{\partial p_y}$$

Hence, $G = x p_y - y p_x = L_z$

Applying what we've recently learned (I.C.T)

$$u \to u_{rot} = u + \delta u \Rightarrow \delta u = [u, L_z]\,d\theta$$

For an arbitrary axis $\hat{n}$, rotation by $d\theta \to \vec{L}\cdot\hat{n}$.
So for a small rotation, and $\vec{F} = \vec{F}[Q_i, P_i]$

$$\vec{F} \Rightarrow \delta\vec{F} = [\vec{F}, \vec{L}\cdot\hat{n}]\,\delta\theta \; ; \quad \delta\vec{F} = \delta\theta\,\hat{n}\times\vec{F}, \text{ so}$$

$$[\vec{F}, \vec{L}\cdot\hat{n}] = \hat{n}\times\vec{F} \Rightarrow [\vec{L}, \vec{L}\cdot\hat{n}] = \hat{n}\times\vec{L}.$$

For example, $[L_x, L_y] = (\hat{y}\times\vec{L})_x = L_z$

$[\vec{L}^2, \vec{L}\cdot\hat{n}] = 0$ since rotation of any scalar like $\vec{L}^2$ leaves it unchanged.