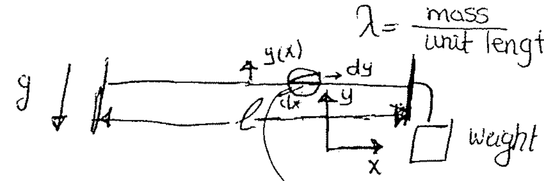
$\boxed{1}$

4-4-2006

Lagrangian Dynamics for ∞ degrees of freedom:-

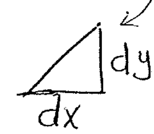
Ex1-

$\lambda = \frac{\text{mass}}{\text{unit length}}$

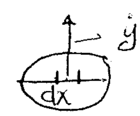


This string is carrying a mass, so it has a tension (T)

$$\text{actual length} = \int_0^l \sqrt{dx^2 + dy^2} = \int_0^l \sqrt{1 + y'^2(x)}\, dx$$



To get the K. E $T = \int_0^l \frac{\dot{y}^2}{2} \lambda \, dx =$



The pot. Energy $=$ (Actual length $- l$) T

$$\text{Actual length} = \int_0^l \left(1 + \frac{y'^2(x)}{2}\right) dx$$

(1st order approximation Taylor series expansion of $\sqrt{1+y'^2}$)

$$= l + \int_0^l \frac{y'^2(x)}{2}\, dx$$

$\Longrightarrow$ So the Lagrangian will be

$$L = \int_0^l \left(\frac{\lambda}{2}\dot{y}^2 - \frac{T}{2} y'^2\right) dx$$

This Lagrangian has problem
- It integrates over whole no. of degrees of freedom
- It contains y' and we don't how to solve y'

at the end of string from both sides ($y = 0$)
we made the following assumption

$$y(x) = \sum_{n=1}^{\infty} a_n \sin k_n x \qquad \text{under condition } k_n l = n\pi \Rightarrow k_n = \frac{n\pi}{l}$$

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$$\dot{y}(x) = \sum_{n=1}^{\infty} \dot{a}_n \sin K_n x \quad , \qquad \text{[using Fourier Theorem]}$$

for $$y' = \sum_{n=1}^{\infty} a_n K_n \cos K_n x$$

So $$L = \frac{\lambda}{2} \int_0^{\ell} \sum_{n,m} \dot{a}_n \dot{a}_m \sin \underbrace{K_n}_{\frac{n\pi}{\ell}} x \, \sin \underbrace{K_m}_{\frac{m\pi}{\ell}} x \, dx$$

$$- \frac{\tau}{2} \int_0^{\ell} \sum_{n,m} a_n K_n a_m K_m \cos K_m x \cos K_m x \, dx$$

Take $\theta = \frac{\pi x}{\ell}$

$$L = \frac{\lambda}{2} \frac{\ell}{\pi} \int_0^{\pi} \sum \dot{a}_n \dot{a}_m \underbrace{\sin n\theta \sin m\theta \, d\theta} - \frac{\tau}{2} \frac{\ell}{\pi} \int_0^{\pi} \sum a_n a_m k_n k_m \underbrace{\cos n\theta \cos m\theta \, d\theta}_{\frac{\pi}{2}}$$

$\rightarrow 0$ if $n \neq m$

$\Rightarrow = \frac{\pi}{2}$ if $n = m$

$$= -\frac{\tau}{2} \frac{\ell}{2} \sum_n a_n^2 K_n^2 + \frac{\lambda \ell}{4} \sum_n \dot{a}_n^2$$

So that the Lagrangian

$$\mathcal{L} = \frac{\lambda \ell}{4} \sum_n \dot{a}_n^2 - \frac{\ell \tau}{4} \sum_n a_n^2 K_n^2$$

(now our new coordinates are a_n and $\dot{a}_n$)

Using ELE $$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}_n} - \frac{\partial \mathcal{L}}{\partial a_n} = 0$$

$$\frac{d}{dt} \, 2 \frac{\lambda \ell}{4} \dot{a}_n + \frac{\ell \tau}{4} 2 a_n K_n^2 = 0$$

$$\Longrightarrow \quad \ddot{a}_n + \frac{\tau}{\lambda} a_n K_n^2 = 0$$

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So the last equ. is a simple harmonic oscillation with ~~amplitude~~ frequency $\frac{\tau K_n^2}{\lambda} = \omega^2$

$$y(x,t) = \sum_n A_n \cos \omega_n t \, \sin K_n x \qquad (1)$$

where $K_n = \frac{n\pi}{\ell}$, $\omega_n = K_n \sqrt{\frac{\tau}{\lambda}} = \frac{n\pi}{\ell}\sqrt{\frac{\tau}{\lambda}}$

This is an equ. of Standing wave which depends on time and position

$$T = \frac{2\pi}{\omega} = \frac{2\ell}{n}\sqrt{\frac{\tau}{\lambda}}$$; $\sqrt{\frac{\tau}{\lambda}} \rightarrow$ This describes the velocity

- if we want to take the gravitational pot. on the string we should add

$$\int_0^\ell y \lambda g \, dx$$ to the pot. energy

from least action principle:-

$$\delta \int_{t_1}^{t_2} L \, dt = 0$$

lets we have a field $\eta(x,t)$

Example Take $\mathcal{L} = \frac{\lambda}{2}\dot{y}^2 - \frac{\tau}{2}y'^2 - \lambda g y$ with "field" $y(x,t)$

so $\cancel{\mathcal{L}}$ $L = \int_{x_1}^{x_2} \mathcal{L}(\eta, \dot{\eta}, \eta', x, t) \, dx$

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$$S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \mathcal{L}(\eta, \dot{\eta}, \eta', x, t)\, dx$$

$$\eta_0 (x,t) \xrightarrow{\delta} \eta_0 + \delta\eta \cdot \alpha \qquad , \delta\eta(x_1) = \delta\eta(x_2) = 0$$

in addition $\delta\eta(x, t_1) = \delta\eta(x, t_2) = 0$

but inside between x_1 and x_2 and (t_1 and t_2) everything can vary

$$0 = \frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left(\frac{\partial \mathcal{L}}{\partial \eta} \delta\eta + \underline{\frac{\partial \mathcal{L}}{\partial \dot{\eta}} \delta\dot{\eta}} + \frac{\partial \mathcal{L}}{\partial \eta'} \delta\eta' \right)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \eta'} \right) \delta\eta$$

$$\int dt \frac{\partial \mathcal{L}}{\partial \dot{\eta}} \delta\dot{\eta} = \frac{\partial \mathcal{L}}{\partial \eta} \delta\eta \Big|_{t_1}^{t_2} - \int \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \eta} \delta\eta$$

$$\int dx \frac{\partial \mathcal{L}}{\partial \eta'} \delta\eta' = \frac{\partial \mathcal{L}}{\partial \eta'} \delta\eta \Big|_{\cancel{x_1}}^{\cancel{x_2}} - \int \frac{d}{d\cancel{x}} \frac{\partial \mathcal{L}}{\partial \eta'} \delta\eta'$$

ELE.
$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\eta}} + \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \eta'} - \frac{\partial \mathcal{L}}{\partial \eta} = 0$$

if we take the gravitational pot. the Lagrangian density

$$\mathcal{L} = \frac{\lambda}{2} \dot{y}^2 - \frac{T}{2} y'^2 - \lambda g y$$

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So ELE $\quad \frac{d}{dt}\lambda\dot{y} - \frac{d}{dx}\tau y' + \lambda g = 0$

$$\Rightarrow \lambda\ddot{y} - \tau y'' + \lambda g = 0$$

Stationary solution when $\ddot{y} = 0$

$$y'' = \frac{\lambda g}{\tau}$$

$$y = y_0 + y_0' x + \frac{1}{2}\frac{\lambda g}{\tau}x^2$$

$y(0) = 0 \Longrightarrow y_0 = 0$

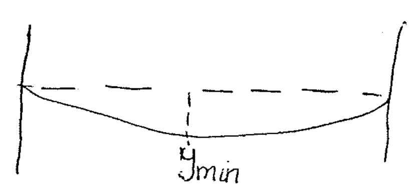
$y(\ell) = 0 \qquad y_0'\ell + \frac{\lambda g}{2\tau}\ell^2 = 0$

$$y_0' = -\frac{\lambda g \ell}{2\tau}$$

To get y_{min} (That is occured at $x = \ell/2$)

$$y_{min} = -\frac{\lambda g \ell^2}{4\tau} + \frac{1}{8}\frac{\lambda g \ell^2}{\tau}$$

$$y_{min} = -\frac{\lambda g \ell^2}{8\tau}$$



Dynamics is

$$\lambda\ddot{y} - \tau y'' = 0$$

The solution will be $y(x,t) = f(Kx - \omega t)$. arbitrary function f

$\ddot{y} = \omega^2 y \quad y'' = K^2 y$ $\quad , \; c = \omega/K$

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so $\lambda \omega^2 - \tau K^2 = 0$

$\Longrightarrow \omega = K\sqrt{\frac{\tau}{\lambda}} \qquad c = \sqrt{\frac{\tau}{\lambda}}$

The same result as (1)

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* To describe a field

$\eta = \eta(\bar{r}, t)$

(1) describe the Lagrangian density

$\mathcal{L}(\eta, \vec{\nabla}\eta, \dot{\eta}, \bar{r}, t)$

(2) L "Lagrangian" $= \iiint d^3\bar{r}\, \mathcal{L}$

(3) $\int_{t_1}^{t_2} L\, dt$ is stationary $= \iiiint dt\, d^3\bar{r}\, \mathcal{L}$

(4) ELE $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}} + \vec{\nabla}\frac{\partial \mathcal{L}}{\partial \vec{\nabla}\eta} - \frac{\partial \mathcal{L}}{\partial \eta} = 0$

$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial\, \partial\eta/\partial x} + \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial\, \partial\eta/\partial y} + \frac{d}{dz}\frac{\partial \mathcal{L}}{\partial\, \partial\eta/\partial z}$

Straightforward application to relativity