

Week 12 Lecture Notes

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Poisson brackets with ICT

Recall that infinitesimal canonical transformations (ICT) take the form

$\vec{\zeta} = \vec{\eta} + d\vec{\eta}$, where $d\vec{\eta} = \epsilon \cdot \frac{d\vec{\eta}}{d\epsilon}$, where ϵ is used to dial the change in $\vec{\eta}$.

Also, recall that $F_2 = \vec{q} \cdot \vec{P}$ is a generator used to get an identity transformation between generalized coordinates. This makes it suitable for use as a generator that incorporates infinitesimal changes. It can be expressed as $F_2 = \vec{q} \cdot \vec{P} + \epsilon G(\vec{q}, \vec{p})$, where G is an infinitesimal transformation generator. Thus, we can express the infinitesimal change with this infinitesimal generator as $d\vec{\eta} = \epsilon \vec{J}_{\vec{\eta}} G$.

We will now see how Poisson brackets behave with ICT. Recall that $[u, v]$ is a number independent of the coordinates used to express u and v . When we use canonical $\vec{\eta}$, these could represent any space/coordinates.

Previously, we had

$[u, v] = (\vec{V}_{\vec{\eta}} u)^T \mathbf{J} (\vec{V}_{\vec{\eta}} v)$. How can we express an infinitesimal change in u ?

We would have the expression

$$\epsilon [u, G] = \epsilon (\vec{V}_{\vec{\eta}} u)^T \mathbf{J} (\vec{V}_{\vec{\eta}} G) = (\vec{V}_{\vec{\eta}} u) d\eta = du(d\vec{\eta})$$

From this, we see how much u will change when the coordinates themselves are changed by G and ϵ .

Example: $G=H$ The Hamiltonian as the infinitesimal generator.

In this example, ϵ becomes dt to show the small increments in time for the system as it progresses with this Hamiltonian. We can now change our previous equation $d\vec{\eta} = \epsilon \vec{J}_{\vec{\eta}} G$ by dividing by ϵ and substituting ϵ and G .

Thus, we have $\frac{d\vec{\eta}}{dt} = \vec{J}_{\vec{\eta}} H$. We can now express this as a Poisson bracket

by again replacing ϵ and G : $dt[u, H] = du(dt) - \frac{\partial u}{\partial t} dt$. This leads to

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}.$$

Now if $\frac{\partial u}{\partial t} = 0$ (meaning there is no explicit dependence on time) AND $[u, H] = 0$, then u is a constant of the motion (meaning u is conserved). (Recall from quantum mechanics that any operator that commutes with the Hamiltonian is conserved.) This also means that $[H, H] = 0$, which leads to $\frac{dH}{dt} = \frac{\partial H}{\partial t}$, will show that energy is conserved if $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$.

We can also use the Poisson bracket with Hamiltonians to express the time dependence of a function $u(t)$, since H describes an infinitesimal translation in time. We first expand $u(t)$ around $t=0$ using the Taylor series.

$$u(t) = u(t=0) + \dot{u}(t=0)t + \frac{\ddot{u}(t=0)t^2}{2!} + \dots$$

Now express the time derivatives as Poisson brackets, and we find that

$$u(t) = u(0) + [u, H] + [\dot{u}, H] + \dots = u(0) + [u, H] + [[u, H], H] + \dots$$

This means that we can use Poisson brackets to solve for $u(t)$.

Example: a particle in free fall

The familiar Hamiltonian for this system is $H = \frac{p^2}{2m} + mgy$. We can solve for $y(t)$ using Poisson brackets to demonstrate that this method will give a familiar result. First, we note that $\vec{\eta} = \begin{pmatrix} y \\ p \end{pmatrix}$, so $\vec{\nabla}_{\vec{\eta}} = \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial p} \hat{p}$.

Referencing the Taylor expansion above, we must first calculate $[y, H]$.

$$\dot{y} = [y, H] = (\vec{\nabla}_{\vec{\eta}} y)^T J(\vec{\nabla}_{\vec{\eta}} H) = (1 \quad 0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} mg \\ p/m \end{pmatrix} = (1 \quad 0) \begin{pmatrix} p/m \\ -mg \end{pmatrix} = \frac{p}{m}$$

Next we calculate $[\dot{y}, H]$:

$$\ddot{y} = [\dot{y}, H] = (\vec{v}_{\vec{\eta}} \dot{y})^T J(\vec{v}_{\vec{\eta}} H) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} mg \\ p \\ -mg \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p \\ -mg \end{pmatrix} = -g$$

Then inserting this into the Taylor expansion, we see that

$y(t) = y(t=0) + \frac{p(t=0)}{m}t - \frac{gt^2}{2}$. We do not need to calculate higher terms in the series because $\ddot{y} = -g$ is constant. All higher derivatives will just go to zero: $\ddot{\ddot{y}} = [\ddot{y}, H] = 0$.

Now that we have explored how a quantity u changes with $G=H$, how would u change if we used other generators? Let us try $G = p_i$. First, we should recall that another definition for the Poisson bracket is

$$[u, G] = \sum_n \frac{\partial u}{\partial q_n} \frac{\partial G}{\partial p_n} - \frac{\partial u}{\partial p_n} \frac{\partial G}{\partial q_n}$$

Then we would have

$$\epsilon[u, G] = \epsilon[u, p_i] = \epsilon \sum_n \frac{\partial u}{\partial q_n} \frac{\partial p_i}{\partial p_n} - \frac{\partial u}{\partial p_n} \frac{\partial p_i}{\partial q_n} = \epsilon \sum_n \frac{\partial u}{\partial q_n} \delta_{i,n} = \epsilon \frac{\partial u}{\partial q_i}$$

Thus, $\epsilon = \delta q_i$, which gives us $\epsilon[u, p_i] = \delta u(\delta q_i)$. This describes behavior for small changes in one coordinate q_i , so p is a generator of displacement.

Example: Cartesian system with r_i and p_i

We could define our generator as $G = \sum_j \vec{p}_j$, which is the center of mass momentum, the total momentum of the system. Then $\delta \vec{r}[u, G] = \delta u(\delta \vec{r})$ describes the displacement of every point by the SAME vector. This can be done actively, with the system itself moving, or passively as the coordinate frame moves around the system.

Let us now try to describe angular momentum in a similar way. Angular momentum is $\vec{L} = \vec{r} \times \vec{p}$. This requires a 6 dimensional phase space with (x, y, z, p_x, p_y, p_z) . For an example, let us consider \vec{L} in the z direction as our generator G . This gives $G = L_z = xp_y - yp_x$. Then $\epsilon[u, L_z]$ gives

$$\begin{aligned}\epsilon[u, L_z] &= \epsilon \sum_n \frac{\partial u}{\partial q_n} \frac{\partial(xp_y - yp_x)}{\partial p_n} - \frac{\partial u}{\partial p_n} \frac{\partial(xp_y - yp_x)}{\partial q_n} = \\ &= \epsilon \left(-\frac{\partial u}{\partial x} y + \frac{\partial u}{\partial y} x - p_y \frac{\partial u}{\partial p_x} + p_x \frac{\partial u}{\partial p_y} \right)\end{aligned}$$

The Poisson bracket has now been separated into coordinate and momentum terms. We can examine the purely coordinate terms to see how L_z has changed them. Expressing those first two terms as matrices, we find that

$$du = \epsilon (\vec{\nabla}_{\vec{r}} u)^T \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (\vec{\nabla}_{\vec{r}} u)^T \begin{bmatrix} dx' \\ dy' \\ dz' \end{bmatrix}$$

That middle matrix is a familiar infinitesimal rotation matrix for $\epsilon = d\alpha$, where $d\alpha$ is an infinitesimally small angle of rotation around the z-axis. Then dx' , dy' , and dz' are infinitesimal changes caused by rotation. From this we can conclude that

$$u' = u + du = u + \vec{\nabla}_{\vec{r}} u \begin{bmatrix} dx' \\ dy' \\ dz' \end{bmatrix} = u_{rot}$$

This means that L_z is the generator of infinitesimal rotations around the z-axis. Similarly, the momentum terms from the Poisson bracket describe how L_z can rotate momentum space if u depends only on p . If u is a mixture of q_i and p_i , L_z will rotate both phase space components of u .

Noether's Theorem

If an ICT generated by G leaves the energy of the system unchanged, then G is conserved (assuming it does not depend on time explicitly).

This means that if $dH = 0$, then $[H, G] = 0$. It follows that $[G, H] = 0$. This also means that $\frac{dG}{dt} = 0$ for $\frac{\partial G}{\partial t} = 0$.

Example: $G = p$

Noether's theorem says that for $G=p$, a system where $[H,p]=0$ will have energy that does not change under displacement. Thus energy is conserved and so is the momentum.

Noether's Theorem can also be used to show symmetry operations that keep the laws of physics invariant.

Continuum Mechanics

Suppose we have a system that has an infinite amount of coordinates. It would be easiest to describe the behavior of this system using Lagrangian mechanics.

Consider a string of length L_0 that is pinned at the ends. This string is given a slight elasticity so that it can slightly increase its length if needed. For example, we could imagine that a pulley is attached to one end of the string with a weight on it. Lifting this mass would release some of the tension on the string, allowing the length to increase. Alternatively, we could imagine a tuner screw from a violin at point $x = L_0$ that has the rest of the string wound around it. Turning the screw can decrease or increase the tension in the string.

The string potential of this elastic string would now be

$$V_{pot}^{string} = (L - L_0)T$$

Where T is the tension in the string, L_0 is the equilibrium length, and L is the stretched or compressed length. This string would have infinite coordinates because every point on the string is independent. When the string is no longer at equilibrium length, that extra length can allow the string to become two dimensional instead of a one-dimensional line in the x - direction. This slight change in height allowed by the change in tension is labeled $\delta y(x)$. Every point along the string could experience a different change in height. However, too much variation in $\delta y(x)$ from point to point could cause the string to rip apart. Therefore, we will impose continuity onto $\delta y(x)$ to keep the string in one piece. This will limit the number of dimensions in our system. Instead of having $\delta y(x)$ values over the whole real axis, it will now be limited to all real integers. This is important because a finite sum over an infinite interval allows us to use the Fourier series to describe $\delta y(x)$.

Recall that the Fourier series is

$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{n=1}^{\infty} b_n \cos(nx)$. The boundary condition of fixed endpoints on the string says that $\delta y(0) = \delta y(L_0) = 0$. Applying this to the Fourier series will give us

$$\delta y(x) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L_0}\right)$$

We would like to express V_{pot}^{string} in terms of a_n , which are NEW generalized coordinates that explain how the string is deformed. Recall that $\sin\left(\frac{n\pi x}{L_0}\right)$ is orthogonal to $\sin\left(\frac{m\pi x}{L_0}\right)$ when $n \neq m$. Therefore, each a_n describes a unique way that the string is deformed at x . The string's deformation at x might change with time, so a_n will also be a function of time.

To proceed with the Lagrangian mechanics, we need to express the potential and kinetic energies as functions of coordinates and velocities. Namely, $V(a_n)$ and $T(\dot{a}_n)$. This process is covered in the next lecture.