April 13th:

$$\begin{pmatrix} ct' \\ \vec{r}' \end{pmatrix} = (L_0)\begin{pmatrix} ct \\ \vec{r} \end{pmatrix}$$ ← 3 parameters: $\beta_x, \beta_y, \beta_z$

L_0 → boost

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{R} & \\ 0 & & & \end{pmatrix}$$

R → rotational matrix

$S \xrightarrow{\vec{\beta}=\frac{\vec{v}}{c}} S'$, no rotation; symmetric

General Lorentz transformation:

$\Rightarrow (L) = (R)(L_0)$ ← 6 parameters

$$L_{0,x} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

L_0 boosts are always symmetric

L ~~boosts~~ transf. are generally not symmetric

Poincaré transformation: $\begin{pmatrix} \text{Boost} \\ \text{Rotation} \\ \text{Offset} \end{pmatrix} \rightarrow \begin{pmatrix} ct' \\ \vec{r}' \end{pmatrix} = (L)\begin{pmatrix} ct \\ \vec{r} \end{pmatrix} + \begin{pmatrix} ct_0 \\ \vec{r}_0 \end{pmatrix}$ ← 10 parameters

$\Rightarrow$ 10 ~~invariants~~ (conserved) quantities: $\vec{p}$, $\vec{E}$, $\vec{L}$, $\vec{R}_{com} - \frac{\vec{P}}{M}t$

$(L_3) = (L_2)(L_1)$: Lorentz transformations form a group! Multiplication, unit $= \mathbb{1}$, inverse $(\vec{\beta} \to -\vec{\beta}$, $R \to R^T)$

$(L_{0_3}) \neq (L_{0_2})(L_{0_1})$ because product is not symmetric

$(R)(L_{0_3}) = (L_{0_2})(L_{0_1})$ two consecutive boosts = 1 boost + 1 rotation

(R) → Thomas precession

How do you determine if a matrix is a Lorentz transformation?

$$\underbrace{(dct)^2 - (d\vec{r})^2 = (ds)^2}_{\text{metric}}$$ introduce metric in Minkowsky space

metric = 2-form → $g(v_1, v_2) = v_1 \cdot v_2$ = Lorentz scalar

v_1, v_2 → 4-vector

$$\begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} = V^\mu \quad , \mu = 0,1,2,3 \; ; \text{or } V^i \; , i = 1,2,3$$ vector

greek latin

$$V_1^0 V_2^0 - V_1^1 V_2^1 - V_1^2 V_2^2 - V_1^3 V_2^3 = g(V_1, V_2)$$ two-form

scalar: S invariant under Lorentz transformation.

$$g_{\mu\nu} V_1^\mu V_2^\nu = (V_1)^T (g)(V_2)$$

uses einstein summation convention

always sum over greek or latin indices (1 up, 1 down), keep remaining indices

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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n-form takes n vectors, and converts them into a scalar

$g(v, \;) = 1$-form

$$= g_{\mu\nu} \underbrace{V^\mu}_{\text{1-vector}} = \underbrace{V_\nu}_{\text{1-form}}$$, $g_{\mu\nu}$ converts them.

Ex: $x^\mu = \begin{pmatrix} ct \\ \vec{r} \end{pmatrix} \rightarrow x_\nu = (ct, -\vec{r})$

2-tensor $\rightarrow$ $g^{\mu\nu} x_\nu = x^\mu$ $\qquad (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ also

So what is g^{μ}_{ν}? A 1-form, 1-tensor

$$g^{\mu}_{\nu} x^{\nu} = x^{\mu} \quad , \text{so } g^{\mu}_{\nu} = \mathbb{1}$$

$$g^{\mu}_{\nu} = g^{\mu\alpha} g_{\alpha\nu}$$

$$\therefore \left(L^{\nu}{}_{\mu}\right)\left(x^{\mu}\right) = \left(x'^{\nu}\right)$$

$$\Rightarrow x'^{\nu} = L^{\nu}{}_{\mu} x^{\mu}$$

$$y'^{\beta} = L^{\beta}{}_{\alpha} y^{\alpha}$$

$$\cancel{g_{\nu\beta} x'^{\nu} y'^{\beta} = g_{\alpha\beta} x^{\alpha} y^{\beta}}$$

want this to be true

$$g_{\nu\beta} x'^{\nu} y'^{\beta} = g_{\nu\beta} L^{\nu}{}_{\mu} x^{\mu} L^{\beta}{}_{\alpha} y^{\alpha} \stackrel{!}{=} g_{\mu\alpha} x^{\mu} y^{\alpha}$$

$$\Rightarrow \underbrace{g_{\nu\beta} L^{\nu}{}_{\mu} L^{\beta}{}_{\alpha} = g_{\mu\alpha}}_{\text{matrix equation}: (L^{T})(g)(L) = (g)}$$

10 independent equations ($\mu > \alpha$ gives nothing new)

$\Rightarrow$ 6 free parameters ($\vec{\beta}$, θ, φ, ψ)

Ex. 1-form

$f(x^{\mu})$ Lorentz scalar

$$\frac{\partial f}{\partial x^{\mu}} = ?$$

$$= \left(\frac{\partial f}{\partial ct}, \frac{\partial f}{\partial x^{1}}, \frac{\partial f}{\partial x^{2}}, \frac{\partial f}{\partial x^{3}}\right) = \nabla_{\mu} f = \partial_{\mu} f$$

$$\sum_{\mu} \partial x^{\mu} \frac{\partial f}{\partial x^{\mu}} = df(x^{\mu} \to x^{\mu} + dx^{\mu}) \Rightarrow \partial_{\nu} f = \text{1-form}$$

because it takes a vector, and makes it a scalar.