Week 2: Equations of motion, Velocity-dependent potentials

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(Participation Project: AMSD Wijerathna)

Recall the *Generalized forces* from the last lecture.

Virtual work is the work done by forces acting on the system on all possible virtual displacements. The infinitesimal virtual work δW due to infinitesimal virtual displacement is,

$$\delta W = \sum_{l=1}^{n} F_l \,.\, \delta r_l = \sum_{l=1}^{n}\sum_{i=1}^{n} F_l . \frac{\partial r_l}{\partial q_i}\delta q_i = \sum_{i=1}^{n} Q_i \delta q_i \tag{1.1}$$

$$\boldsymbol{Q_i} = \sum_{l=1}^{n} \boldsymbol{F_l} . \frac{\boldsymbol{\partial r_l}}{\boldsymbol{\partial q_i}} \tag{1.2}$$

The coefficient Q_i is called the i^{th} *component of the generalized force associated with generalized coordinate.*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{1.3}$$

The Equation (1.3) is known as *Lagrange's equations of motion*. Since *Lagrange's equations of motion* are derived from the Newton's equations of motion, they do not represent a new physical theorem but merely express the same laws of motion in a different way. In Lagrangian formulation, the equations of motion are obtained entirely in terms of scalar operations in the configuration space. Therefore, they have the same form in all coordinate systems and represent a uniform way of writing the equations of motion independent of coordinates used.

Recall the *Euler-Lagrange Equation* of Motion from the last lecture.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{1.4}$$

where q_k's are generalized co-ordinates while $\dot{q}_k$'s are generalized velocities. The *Langrangian* (L) is a combination of potential and kinetic energies as follows.

$$L = T - V \tag{1.5}$$

Let's now do an example to understand the application of *Lagrange's Equation of motion* and the physics behind the result.

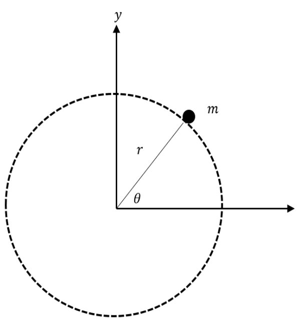

Figure 1. 1

Example: Consider a string with a mass m on the end and the starting angle as θ from the x direction (see Fig. 1.1). Assume that the motion takes place in a horizontal plane, obtain the *Lagrange's Equation of motion* for r and θ.

Solution: The position co-ordinates of mass m,

$$x = \cos\theta \text{ and } y = \sin\theta$$

The kinetic energy has radial and tangential parts.

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{1.6}$$

The *Generalized force* in the r direction,

$$Q_r = F_x \frac{\partial x}{\partial r} + F_y \frac{\partial y}{\partial r} = F_x \cos\theta + F_y \sin\theta = \vec{F}\,.\hat{r} \tag{1.7}$$

The *Generalized force* in the θ direction,

$$Q_r = F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} = rF_y \cos\theta - rF_x \sin\theta = -yF_x + xF_y \tag{1.7}$$

$$Q_r = \vec{r} \times \vec{F}\ (Torque) \tag{1.8}$$

The *Lagrange's Equation of motion* in radial direction,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r \qquad m\ddot{r} - mr\dot{\theta}^2 = Q_r = \vec{F}\,.\hat{r}$$

$$m\ddot{r} = mr\dot{\theta}^2 + \vec{F}\,.\hat{r} \tag{1.9}$$

The *Lagrange's Equation of motion* in tangential direction,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta \qquad \frac{d(mr^2\dot{\theta})}{dt} = Q_r = \vec{r} \times \vec{F}\ (Torque) \tag{2.0}$$

$$m\ddot{\theta}r^2 + 2r\dot{r}m\dot{\theta} = Torque \tag{2.1}$$

Equation (1.9) is the application of Newton's Second Law in radial direction and it is complete with the centrifugal force, $mr\dot{\theta}^2$ which give us to work in a rotating reference frame. The Equation (2.0) indicates that torque is equal to the rate of change of angular momentum. The Equation (2.1) deals with the tangential Newton's Second Law. The term, $2mr\dot{r}\dot{\theta}$ is the Coriolis force which we can deal with rotating frames.

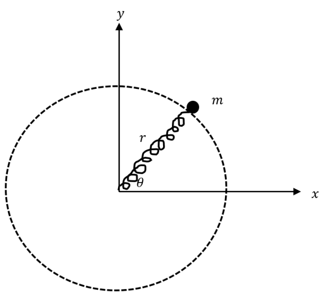

Figure 1. 2

Example (Circular Motion of a Spring): Consider a spring (spring constant is k) with a mass m on the end (see Figure 1.2). The equilibrium length of the spring is l. Let the spring have length $l + r$, and its angle with x- axis be θ. Assuming that the motion takes place in a horizontal plane, find the equation of motion for r and θ using *Euler-Lagrange Equation.*

Solution: The kinetic energy has radial and tangential parts.

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{2.2}$$

The potential energy is only the spring energy.

$$V(r) = \frac{1}{2}k(r-l)^2 \tag{2.3}$$

$$L \equiv T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}k(r-l)^2 \tag{2.4}$$

The *Lagrangian* has two variables, r and θ. Therefore, there is two *Euler-Lagrange Equations.*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \qquad m\ddot{r} = mr\dot{\theta}^2 - k(r-l) \tag{2.5}$$

and,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \qquad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \tag{2.6}$$

$$mr^2\ddot{\theta} + 2m\dot{r}r\dot{\theta} = 0 \tag{2.7}$$

The Equation (2.6) indicates the angular momentum is conserved.

Furthermore Explanation: *What happened if the solution is stationary?*

Then, r and $\dot{\theta}$ are constant values $(r_0\,, \omega_0)$. According to the Equation (2.5), we can show that,

$$mr_0{\omega_0}^2 = k(r_0 - l) \tag{2.8}$$

$${\omega_0}^2 = \frac{k(r_0 - l)}{mr_0}; r_0 > l \tag{2.9}$$

1.1 Small Oscillations around Stationary Solutions

To find the frequency of small oscillations about the circular motion, we need to introduce a new generalized co-ordinate.

Let $r(t) = r_0 + \delta r$ where δr is very small (more precisely, $\delta r \ll r_0$) and assume $\dot{\theta} = \omega_0 + \delta\dot{\theta}$.

The fact is circular motion occurs when, $\dot{r} = \ddot{r} = 0$. Therefore, Equation (2.5) can be written as,

$$m\delta\ddot{r} = m(r_0 + \delta r)(\omega_0 + \delta\dot{\theta})^2 - k[(r_0 + \delta r) - l] \tag{3.0}$$

The terms not involving δ on the right-hand side cancel, due to the definition of the equilibrium point.

$$mr_0\omega_0 = k(r_0 - l) \tag{3.1}$$

Considering the Equation (2.6), $mr^2\dot{\theta}$ is a constant.

$$mr^2\dot{\theta} = m{r_0}^2\omega_0 \tag{3.2}$$

$$m(r_0 + \delta r)^2(\omega_0 + \delta\dot{\theta}) = m{r_0}^2\omega_0 \tag{3.3}$$

$$\delta\dot{\theta} \approx -\frac{2\omega_0}{r_0}\delta r\,; (\delta\dot{\theta}\delta r \;\approx 0) \tag{3.4}$$

The Equations (3.0), (3.1), and (3.4) imply that,

$$m\delta\ddot{r} \approx m{\omega_0}^2\delta r - 4m{\omega_0}^2\delta r - k\delta r\,; (\delta\dot{\theta}^2 \approx 0) \tag{3.5}$$

$$\delta\ddot{r} \approx -\left(3{\omega_0}^2 + \frac{k}{m}\right)\delta r \tag{3.6}$$

This is a simple-harmonic-oscillator equation in the variable δr. Therefore, the frequency of small oscillations about a circle is,

$$\omega \approx \sqrt{\left(3{\omega_0}^2 + \frac{k}{m}\right)} \tag{3.7}$$

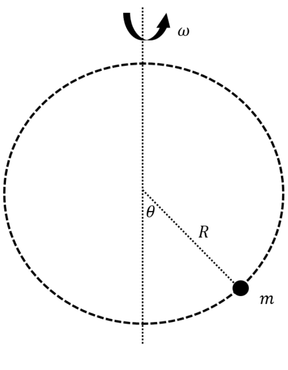

Figure 1. 3

Example (Bead on a rotating hoop): A bead is free to slide on along a frictionless hoop of radius R. The hoop rotates with a constant angular speed ω around a vertical diameter (see Figure 1.3). Find the equation of motion.

Solution:

$$L = \frac{1}{2}m\left(\omega^2 R^2 (\sin\theta)^2 + R^2\dot{\theta}^2\right) + mgR\cos\theta \tag{3.8}$$

The equation of motion is,

$$R\ddot{\theta} = \sin\theta\,(\omega^2 R\cos\theta - g) \tag{3.9}$$

Special Cases:

(a) $\omega_0 = 0$ and $\sin\theta \approx \theta$

$$\ddot{\theta} \approx -\frac{g}{R}\theta \tag{4.0}$$

According to the Equation (4.0), bead undergoes a simple harmonic oscillation with a frequency, $\sqrt{\frac{g}{R}}$.

(b) In equilibrium position,

Equilibrium occurs when $\dot{\theta} = \ddot{\theta} = 0$. If $\omega^2 \geq \frac{g}{R}$, then $\theta = 0, \theta = \pi$ and $\cos\theta_0 \equiv \frac{g}{\omega^2 R}$ are all equilibrium points. But $\theta = 0, \theta = \pi$ cases are unstable. Therefore, $\cos\theta_0 \equiv \frac{g}{\omega^2 R}$ is the only stable equilibrium. $\cos\theta_0$ never becomes negative. Therefore, bead always in the lower half of the circle.

(c) The frequency of small oscillations

Let $\theta = \theta_0 + \delta\theta$ in Equation (3.9) and expand to first order in $\delta\theta$. Using $\cos\theta_0 \equiv \frac{g}{\omega^2 R}$,

$$\delta\ddot{\theta} + \omega^2 (\sin\theta_0)^2\,\delta\theta = 0 \tag{4.1}$$

The frequency of small oscillations is,

$$\omega\sin\theta_0 = \sqrt{\omega^2 - \frac{g^2}{\omega^2 R^2}} \tag{4.2}$$

(d) Forces of Constraint

In here we have two force of constraints, as we have two constraints $(r, \dot{\theta})$.

Finding force of constraint in radial direction

To find it, we need to look at what happens to r constraint if we perturb it slightly from its value R. Therefore,

$$R: R + \delta r \tag{4.3}$$

Then kinetic energy expression becomes more complicated.

$$T = \frac{m}{2}\left\{\delta\dot{r}^2 + (R^2 + 2R\delta r)[\dot{\theta}^2 + (\sin\theta)^2\omega^2]\right\} \tag{4.4}$$

The potential energy can be expressed as follows including the centripetal force F_r.

$$V = mg(R + \delta r)\cos\theta + F_r\delta r \tag{4.5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial(\delta\dot{r})}\right) = m\delta\ddot{r} \tag{4.6}$$

$$\frac{\partial L}{\partial(\delta r)} = mR\left[\dot{\theta}^2 + (\sin\theta)^2\omega^2\right] - mg\cos\theta - F_r \tag{4.7}$$

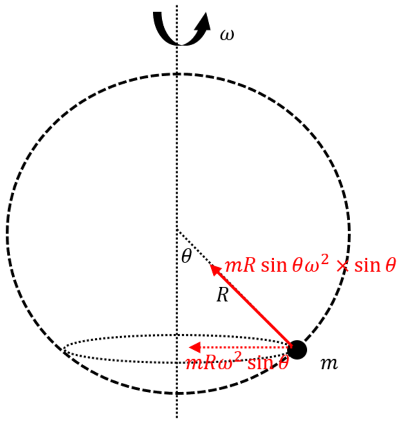

Figure 1. 4

The Equation (4.6) must be zero as force of constraint prevent any motion in the radial direction. Therefore, $\dot{r} = \ddot{r} = 0$. It implies that centripetal force of constraint is,

$$F_r = mR\left[\dot{\theta}^2 + (\sin\theta)^2\omega^2\right] - mg\cos\theta \tag{4.8}$$

Here $mR\dot{\theta}^2$ is a centripetal force. If $\dot{\theta} = 0$, then $mR\sin\theta\omega^2 \times \sin\theta$ is also centripetal force (see Figure 1.4). $-mg\cos\theta$ is due to the gravity towards radial direction.

Finding force of constraint in φ direction

Considering $\omega: \omega + \delta\dot{\varphi}$,

$$T = \frac{m}{2}\left\{R^2\dot{\theta}^2 + R^2(\sin\theta)^2(\omega + \delta\dot{\varphi})^2\right\} \tag{4.9}$$

The potential energy can be expressed as follows including a torque term.

$$V = mg\cos\theta + (Torque)\delta\varphi \tag{5.0}$$

$$V = mg\cos\theta + R\sin\theta F_\varphi\,\delta\varphi \tag{5.1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial(\delta\dot{\varphi})}\right) = mR^2(\sin\theta)^2\delta\ddot{\varphi} + 2mR^2\sin\theta\cos\theta(\omega + \delta\dot{\varphi})\,\dot{\theta} \tag{5.2}$$

$$\frac{\partial L}{\partial(\delta\theta)} = -R\sin\theta F_\theta \tag{5.3}$$

$\delta\ddot{\varphi} = \delta\dot{\varphi} = 0$, because φ is the constraint.

$$F_\varphi = -2mR\cos\theta\,\dot{\theta}\omega \tag{5.4}$$

Equation (5.4) represents the Coriolis Force corresponding to the rotation of frames.

1.2 The Lagrangian for Electromagnetic Field

The electromagnetic forces on a moving charge are non-conservative as these forces depend on the velocity. The force on a particle of charge $\boldsymbol{q}$ moving with velocity $\boldsymbol{v}$ in an electromagnetic field is given by *Lorentz force*.

$$\vec{F} = q(\vec{E} + \vec{V} \times \vec{B}) \tag{5.5}$$

Electric field can be represented in scalar potential $\emptyset$ and vector potential $\vec{A}$ as,

$$\vec{E} = -\vec{\nabla}\emptyset - \frac{\partial\vec{A}}{\partial t} \tag{5.6}$$

Potential energy depends upon co-ordinates. Therefore, *Lagrangian* can be expressed as follows.

$$L = \frac{m}{2}\sum_j \dot{r}_j - q\emptyset(\vec{r},t) + q\sum_j A_j\dot{r}_j \tag{5.7}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = \frac{d}{dt}(m\dot{r}_i + qA_i) \tag{5.8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = m\ddot{r}_i + q\frac{\partial A_i}{\partial t} + q\sum_j \frac{\partial A_i}{\partial r_j}\dot{r}_j \tag{5.9}$$

$$\frac{\partial L}{\partial r_i} = -q\frac{\partial \emptyset}{\partial r_i} + q\sum_j \frac{\partial A_j}{\partial r_i}\dot{r}_j \tag{6.0}$$

We know that, $\vec{B} = \vec{\nabla} \times \vec{A}(\vec{r},t)$. Therefore, $q(\vec{v} \times \vec{B}) = \dot{\vec{r}} \times (\vec{\nabla} \times q\vec{A})$.

A long and tedious calculation shows that

$$\left[\dot{\vec{r}} \times (\vec{\nabla} \times q\vec{A})\right]_i = q\sum_j \frac{\partial A_j}{\partial r_i}\dot{r}_j - q\sum_j \frac{\partial A_i}{\partial r_j}\dot{r}_j \tag{6.2}$$

We know that, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = \frac{\partial L}{\partial r_i}$. Therefore, considering the Equations (5.6), (5.8), (5.9), and (6.2) we can conclude that,

$$\boldsymbol{F_{Lorentz} = m\ddot{r}_i = \left(q\vec{E}\right)_i + q\left(\vec{v} \times \vec{B}\right)_i} \tag{6.3}$$