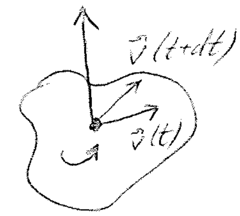
Rotating Objects

$$d_{space}\vec V = d_{body}\vec V + d_{rot}\vec V$$

$d_{body}\vec V$: may be 0

$d_{rot}\vec V$: $d\vec\Omega \times \vec V \equiv d\Omega\,(\hat n \times \vec V)$



$\vec V$ changes relative to rigid object
(coord system is fixed on rigid object)

R is matrix which describes transition from fixed to the body system

$$(\vec V)' = R(\vec V)$$

t: $\quad (\vec V) = R^T(\vec V)'$

$t+dt$?: $\quad \cancel{(\vec V + d_{sp}\vec V) = R^T(\vec V + d\vec V)' + [R^T(t+dt) - R(t)](\vec V)'}$

$$d(\vec V) = R^T d(\vec V)' + dR^T(\vec V)'$$

t: $\quad R(t)$

$t+dt$: $R(t+dt)$ it's diff. because coord. system has been actively rotated around axis with small angle $d\Omega$

$$R(t+dt) = \left(\mathbb{1} - \Omega(\hat n \cdot \vec M)\right)R$$
← this is expressing rotation relative already rotated system

$$R^T + dR^T = R^T(t+dt) = R^T\left(\mathbb{1} + d\Omega(\hat n \cdot \vec M)\right)$$
rotation in the body system

$\underline{M_n^T = -M_n}$

$$dR^T = R^T d\Omega(\hat n \cdot \vec M)$$

$$\Rightarrow \quad d(\vec V) = R^T\left[d(\vec V)' + (d\Omega\,\hat n \cdot \vec M)(\vec V)'\right]$$

rotation relative unrotated system

$$R\, dR^T R^T = R R^T d\Omega (\hat{n} \vec{M}) R^T = d\Omega (\hat{n} \vec{M}) R^T$$

$$\Rightarrow d(\vec{V}) = R^T d(\vec{V})' + d\Omega (\hat{n} \vec{M})(\vec{V})$$ (in unrotated or unprimed system)

$$\left(\frac{d}{dt}\right)_{space} \{\vec{V}\} = \overset{(1)}{\left(\frac{d}{dt}\right)_{body} \{\vec{V}\}} + \overset{(2)}{\vec{\omega} \times \vec{V}} \quad , \quad d\vec{\Omega} = \vec{\omega}\, dt$$

this is true in any coord. system

$$\vec{\tau} = \frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L}$$

1) in space system: $(2)_e = \sum_{m,n} \varepsilon_{emn} (\vec{\omega})_m (\vec{V})_n$

2) in rot.-ing body system: $(2)'_e = \sum \varepsilon_{emn} (\vec{\omega})'_m (\vec{V})'_n$

$$R = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Can we express $\underset{(\vec{\omega})'}{\vec{\omega}}$ in primed system using this angles?

If we change $\psi \Rightarrow$ we're rotating the final axes $\Rightarrow$ change of ψ has only z' comp.

If we change θ = its change will have only x component
but need to apply last rotation around ψ to go into ' system

We get

$$(\vec{\omega})' = \begin{pmatrix} \cos\psi\, \dot{\theta} + \sin\psi \sin\theta\, \dot{\phi} \\ -\sin\psi\, \dot{\theta} + \cos\psi \sin\theta\, \dot{\phi} \\ \dot{\psi} + \cos\theta\, \dot{\phi} \end{pmatrix}$$

If we want to change ψ, we rotate around old z axis

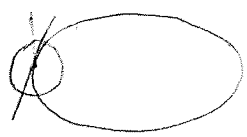
Example

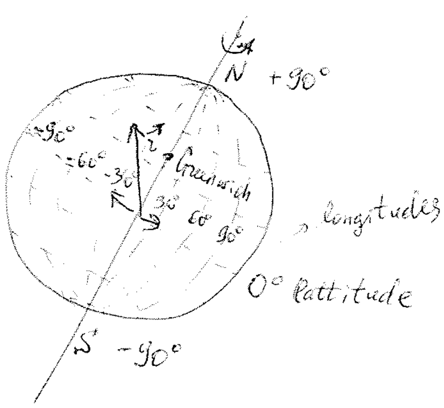
Earth rotates in elliptic path, and its axis makes some angle with ellipse normal axis. ~~But~~ the angle changes during the time, but we assume its fixed, because changes very slowly. $\Rightarrow \dot{\theta} = 0, \dot{\phi} = 0$

$\theta \approx 23^\circ$

$\dot{\psi} = \frac{366.25}{365.25} \cdot \frac{2\pi}{24 \cdot 3600\,s}$ — rot. speed around earth's axis

$\dot{\phi} \approx \frac{2\pi}{365.25 \cdot 3600 \cdot 24\,s}$ relative to fixed x axis from sun to the Earth





$Lat = 90^\circ - \vartheta \quad : \quad \vartheta = 90^\circ - Lat$

$\varphi = Lon + \dot{\psi} \cdot t$

With Earth centered coord. system; $\left(\vec{r}_{Earth}\right)' = \underbrace{r}_{R_{Earth}}\big(\cos(Lat)\cos(Lon+\dot{\psi}t),\ \cos(Lat)\sin(Lon+\dot{\psi}t),\ \sin(Lat)\big)$

radius-vector for some ~~point~~ point on the surface of the Earth

In absolute system of coord.-s (rel. to stars) the velocity will be;

$$\vec{V}_{space} = V_{Earth} + \vec{\omega} \times \vec{r}_{Earth}$$

At North pole $\vec{\omega} \times \vec{r}_{Earth} = 0$

and acceleration;

$$\cancel{\vec{a}_{space} = \vec{a}_{Earth} + \vec{\omega} \times \vec{V}_{Earth}}$$

$$\frac{d_{space} \vec{V}_{space}}{dt} = \vec{a}_{space} = \frac{d_{Earth} \vec{V}_{space}}{dt} + \vec{\omega} \times \vec{V}_{space} =$$

$$= \frac{d_{Earth} \vec{V}_{Earth}}{dt} + \vec{\omega} \times \frac{d \vec{r}_{Earth}}{dt} + \vec{\omega} \times \vec{V}_{Earth} + \vec{\omega} \times \vec{\omega} \times \vec{r}_{Earth} \Rightarrow$$

$$\frac{\vec{F}}{m} = \vec{a}_{Earth} + \underbrace{2 \vec{\omega} \times \vec{V}_{Earth} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{Earth})}_{\text{pseudoforces}}$$

$$\uparrow V_{tot}^2 = (r \cdot \cos(Lat) \dot{\varphi} + V_E)^2$$

$-\vec{\omega} \times (\vec{\omega} \times \vec{r}_{Earth})$ is in plane of $\vec{\omega}$ and $\vec{r}$

$-\vec{\omega} \times (\vec{\omega} \times \vec{r}_{Earth}) = -\omega^2 r \cos(Lat) \to$ centrifugal ~~centripetal~~ force

$-2 \vec{\omega} \times \vec{V}_{Earth} = 2 \vec{V}_{Earth} \times \vec{\omega} \to$ 2nd part of centrifugal force. - Coreolis force
(if you go to East ⇒ acc.-n ⊙)

$$\Delta X = \omega V t^2 \sin(Lat) = \omega t \Delta S \sin(Lat)$$

For hurricanes