02·14·06 Classical Mechanics

$\mathbb{I}$, $\vec{L} = \mathbb{I}\cdot\vec{\omega}$, $T = \frac{1}{2}\vec{\omega}\cdot\mathbb{I}\cdot\vec{\omega} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

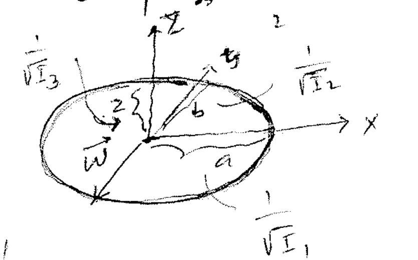
$$(\mathbb{I})' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \qquad (\mathbb{I})' = R(t)(\mathbb{I})\,R^T(t)$$

$$\vec{\rho}\cdot\mathbb{I}\cdot\vec{\rho} = 1 = \rho^2(\hat{\rho}\cdot\mathbb{I}\cdot\hat{\rho}) = \rho^2 I_\rho$$

1-2-3 system;

$$I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2 = 1 \qquad (\vec{\rho})' = (\rho_1, \rho_2, \rho_3)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{Ellipsoid}$$



3 half-axes

$$\frac{1}{\sqrt{I_1}},\ \frac{1}{\sqrt{I_2}},\ \frac{1}{\sqrt{I_3}}$$

Free rotation $\rightarrow$

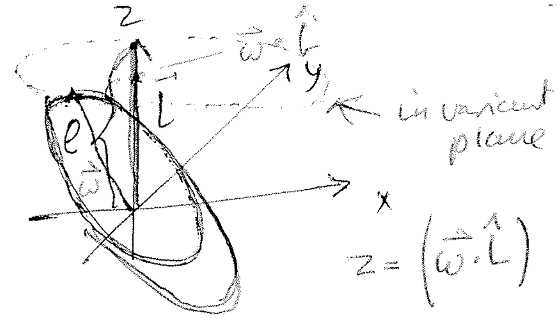
T conserved, $\vec{L}$ conserved

$$2T = \vec{\omega}\cdot\mathbb{I}\cdot\vec{\omega} = \text{const.}$$

$$\frac{\omega_1^2 I_1}{2T} + \frac{\omega_2^2 I_2}{2T} + \frac{\omega_3^2 I_3}{2T} = 1$$

half-axes in ω-space $\quad \sqrt{\frac{2T}{I_1}},\ \sqrt{\frac{2T}{I_2}},\ \sqrt{\frac{2T}{I_3}}$

$$\vec{\omega}\cdot\vec{L} = 2T = (\vec{\omega}\cdot\hat{L})\, l$$



$$\omega_L = \frac{2T}{l}$$

$$\nabla_{\vec{\omega}} \frac{1}{2T}(\vec{\omega}\cdot I\cdot\vec{\omega}) = \frac{1}{2T}2(\mathbb{I}\cdot\vec{\omega}) = \frac{\vec{L}}{T}$$ normal on ellipsoid surface = constant = normal on invariant plane

1-2-3 system: $\omega_1 = \dfrac{L_1}{I_1}$

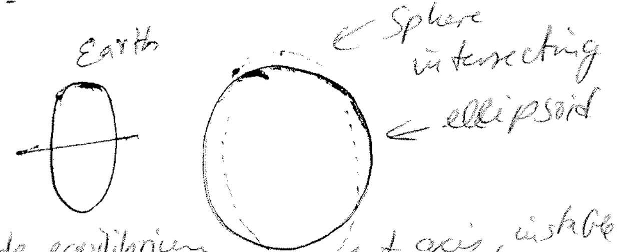
$$\frac{I}{2T}\left(\frac{L_1}{I_1}\right)^2 + \frac{I_2}{2T}\left(\frac{L_2}{I_2}\right)^2 + \frac{I_3}{2T}\left(\frac{L_3}{I_3}\right)^2 = 1$$

new ellipsoid ("Binet"), half axes $\sqrt{2TI_1}, \sqrt{2TI_2}, \sqrt{2TI_3}$

$$\cancel{\omega_1^2\frac{I_1}{2T} + \omega_2^2\frac{I_2}{2T} + \omega_3^2\frac{I_3}{2T} = 1}$$

$$L_1^2 + L_2^2 + L_3^2 = \vec{L}^2$$

$$\frac{L_1^2}{2TI_1} + \frac{L_2^2}{2TI_2} + \frac{L_3^2}{2TI_3} = 1$$



Stable equilibrium around largest and shortest axis, unstable 3rd

$$\vec{N} = \frac{d_{space}\vec{L}}{dt} = \frac{d_{body}\vec{L}}{dt} + \vec{\omega}\times\vec{L} \qquad \vec{L} = (\omega_1 I_1, \omega_2 I_2, \omega_3 I_3)$$

$$N_1 = I_1\dot{\omega}_1 + \omega_2 L_3 - \omega_3 L_2 = \omega_2\omega_3 I_3 - \omega_3\omega_2 I_2 = I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3)$$

$$N_2 = I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1)$$

$$N_3 = I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2)$$

$\vec{N} = 0$

$I_1 = I_2 = I_\perp$

$\rightarrow \omega_3 = \text{const.}$

$$I_\perp \dot{w}_1 = w_2 w_3 (I_\perp - I_3)$$

$$I_\perp \dot{w}_2 = w_3 w_1 (I_3 - I_\perp)$$

$$\ddot{w}_1 = \dot{w}_2 \left(1 - \frac{I_3}{I_1}\right) w_3 \qquad \dot{w}_2 = -w_1 \left(1 - \frac{I_3}{I_1}\right) w_3$$

$$= -w_1 \left(1 - \frac{I_3}{I_\perp}\right)^2 w_3^2$$

$$w_1 = A \cos(\Omega t + \varphi_0) \qquad \Omega = \pm \left(1 - \frac{I_3}{I_\perp}\right) w_3$$

$$\dot{w}_2 = -A\Omega \cos(\Omega t + \varphi_0) = -\omega_1 \Omega$$

$$w_2 = -A \sin(\Omega t + \varphi_0)$$

$$|\vec{\omega}| = \omega_1^2 + \omega_2^2 + \omega_3^2 = \sqrt{A^2 + \omega_3^2} = \text{const.}$$

$$\omega_3 = \cos\theta\, |\vec{\omega}| \Rightarrow \cos\theta = \frac{\omega_3}{\sqrt{A^2 + \omega_3^2}} = \text{const.}$$

$\vec{\omega}$ is precessing on the body cone (opening angle θ) with constant precession frequency Ω.

$$|\vec{L}|^2 = I_\perp^2 A^2 + I_3^2 \omega_3^2$$

$$2T = I_\perp(\omega_1^2 + \omega_2^2) + I_3 \omega_3^2 = I_\perp A^2 + I_3 \omega^2$$

solve for A, ω_3.