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$$\mathcal{L} = \frac{1}{2}(\dot{\vec{q}})^T (T(\vec{q},t))(\dot{\vec{q}}) + (\dot{\vec{q}})^T(\vec{a}(\vec{q},t)) + \mathcal{L}_0(\vec{q},t)$$

E.M. example

$$\mathcal{L} = \frac{m}{2}\vec{v}^2 - \underset{\text{charge}}{e}\,\Phi(\vec{r},t) + \underset{\text{charge}}{e}\,\vec{v}\cdot\vec{A}(\vec{r},t)$$

$$(\vec{q}) \to (\vec{r}) \qquad (\dot{\vec{q}}) \to (\vec{v}) \qquad T = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$\mathcal{L}_0 = -e\Phi(\vec{r},t) \qquad (\vec{a}) = (e\vec{A}(\vec{r},t))$$

$$(\vec{p}) = (T)(\vec{v}) + (e\vec{A}) \qquad \vec{p} = m\vec{v} + e\vec{A}$$ canonical (NOT ordinary) momentum

$$H = \frac{1}{2}(\vec{p} - e\vec{A})^T (T)^{-1} (\vec{p} - e\vec{A}) + e\Phi(\vec{r},t)$$

$$= \underbrace{\frac{(\vec{p} - e\vec{A})^2}{2m}}_{T} + \underbrace{e\Phi}_{V}$$

Here $H = E$ even though we have a linear term

Is E conserved? Only if A or Φ are not time dependent

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$$\frac{\partial H}{\partial p_i} = \dot{r}_i \implies \vec{v} = \frac{\vec{p} - e\vec{A}}{m}$$

$$\frac{\partial H}{\partial r_i} = -\dot{p}_i$$ what depends on r?

$$\frac{(\vec{p} - e\vec{A})}{m} \cdot \left(-e\frac{\partial \vec{A}}{\partial r_i}\right) + e\frac{\partial \Phi}{\partial r_i} = -m\dot{v}_i - e\frac{dA_i}{dt}$$

$$= -m\dot{v}_i - e\frac{\partial A_i}{\partial t} - \sum_j \frac{\partial A_i}{\partial r_j} v_j$$

Sort terms

$$m\dot{v}_i = \underbrace{\left(-e\frac{\partial A_i}{\partial t} - \frac{\partial \Phi}{\partial r_i}\right)}_{e(\vec{E}_i)} + \underbrace{e\vec{v}\frac{\partial \vec{A}}{\partial r_i} - e\sum_j \frac{\partial A_i}{\partial r_j} v_j}$$

$$\underbrace{+ e\left(\sum_j v_j \underbrace{\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)}_{\substack{\text{(magnetic field)} \sum_k \varepsilon_{ijk} B_k \\ \vec{B} = \vec{\nabla} \times \vec{A} \\ = \text{rot } \vec{A}}}\right)}_{e(\vec{v} \times \vec{B}_i)}$$

$$\boxed{m\dot{\vec{v}} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

Pops out again. Once from $\mathcal{L}$

* From C.M. → Q.M., kinematic variables get replaced by operators

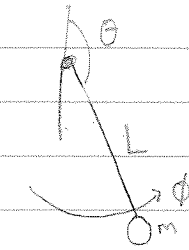
Q.M.:

$$p_i \to \frac{\hbar}{i}\frac{\partial}{\partial q_i} \qquad \vec{q} \to \psi(\vec{q}, t)$$

$$H \to i\hbar\frac{\partial}{\partial t} \qquad i\hbar\frac{\partial \psi}{\partial t} = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - e\vec{A}\right)^2 + e\Phi\right]\psi$$

Conserved momenta:

Pendulum



$$\left(\vec{q}\right) = \begin{pmatrix}\theta \\ \phi\end{pmatrix} \qquad \left(\dot{\vec{q}}\right) = \begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2}\left(\dot{\vec{q}}\right)^T \begin{pmatrix} mL^2 & 0 \\ 0 & m\sin^2\theta L^2 \end{pmatrix} \left(\dot{\vec{q}}\right) - mgL\cos\theta$$

$$\left(\vec{p}\right) = \begin{pmatrix} mL^2 & 0 \\ 0 & m\sin^2\theta L^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

$$H = \frac{1}{2}\left(\vec{p}\right)^T \begin{pmatrix} \frac{1}{mL^2} & 0 \\ 0 & \frac{1}{mL^2\sin^2\theta} \end{pmatrix} \left(\vec{p}\right) + mgL\cos\theta$$

$$= \frac{p_\theta^2}{2mL^2} + \underbrace{\frac{p_\phi^2}{2mL^2\sin^2\theta} + mgL\cos\theta}_{V_{eff}}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \qquad p_\phi = \text{const}$$

can be integrated once $\theta(t)$ is known

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mL^2\sin^2\theta} \qquad \dot{\theta} = \frac{p_\theta}{mL^2}$$

plug in

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{mL^2\sin^3\theta} + mgL\sin\theta \quad \Rightarrow 2^{nd} \text{ order Diff Eq. for } \theta$$

$$\vec{q} \to \vec{q}' = \vec{q} + \vec{\alpha}\, ds \qquad \vec{\alpha} = \frac{d\vec{q}}{ds} \qquad \text{e.g. } \vec{q} = \vec{r} \Rightarrow \alpha = \widehat{ds}$$

ds is displacement of all objects in one specific direction $\widehat{ds}$

$$dH(ds) \overset{\text{we want}}{=} 0 = \sum_i \frac{\partial H}{\partial q_i} dq_i = \underbrace{\left(\sum_i -\dot{p}_i \alpha_i\right)}_{=0} ds$$

invariant under displacement

IF H is invariant under such displacement (depends only on relative positions)

$$\frac{d}{dt}\left(\sum_i p_i \alpha_i\right) = 0$$

COM momentum in direction of displacement of all particles

$$(\vec{q}) = \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_N \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} \frac{d\vec{r}_1}{ds} \\ \vdots \\ \frac{d\vec{r}_N}{ds} \end{pmatrix}$$

$$0 = \frac{d}{dt} \sum_{i=1}^{N} \frac{d\vec{r}_i}{ds} \vec{p}_i \quad \left(= \frac{d}{dt} \sum_{i=1}^{N} d\hat{s} \cdot \vec{p}_i \right.$$

center of mass momenta is conserved in direction $\hat{ds}$

Hamilton's Principle

$\int_{t_1}^{t_2} \mathcal{L}\, dt$ is stationary $q_1 \ldots q_k$

$$H = \sum_i p_i \dot{q}_i - \mathcal{L} \qquad \mathcal{L} = \sum_i p_i \dot{q}_i - H$$

$$\underbrace{\int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H\right) dt}_{f(p_i, q_i, \dot{q}_i, \dot{p}_i, t)} \quad \text{is stationary}$$

aside $\left[= \int_{\vec{q}(t_1)}^{q(t_2)} \sum p_i \, dq_i - \int_{t_1}^{t_2} H\, dt \right]$

Now it becomes an integration of a path in phase space

$$\delta \int = \int \left(\sum \frac{\partial f}{\partial q_i} \delta q_i + \frac{\partial f}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial f}{\partial p_i} \delta p_i + \frac{\partial f}{\partial \dot{p}_i} \delta \dot{p}_i \right) = 0$$

$\Rightarrow$ "ELE"

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{q}_i} - \frac{\partial f}{\partial q_i} = 0 \quad \Rightarrow \quad \dot{p}_i + \frac{\partial H}{\partial q_i} = 0$$

$$\underbrace{\cancel{\frac{d}{dt}\frac{\partial f}{\partial \dot{p}_i}}}_{0} - \frac{\partial f}{\partial p_i} = 0 \quad \Rightarrow \quad -\dot{q}_i - \frac{\partial H}{\partial p_i} = 0$$