PROBLEM 1

The eigenstates for a SINGLE neutron are

$$\psi_n^{\pm}(x) = A_n \sin\left(\frac{nx}{L}\right), \quad n = 1, 2, \dots \quad \text{with } E_n = \frac{\hbar^2 n^2}{2mL^2}$$

where $\psi^+(x) = \psi(x) \otimes |\uparrow\rangle_{spin}$ and $\psi^-(x) = \psi(x) \otimes |\downarrow\rangle_{spin}$

Since the spin doesn't influence the energy eigenvalues, we can factorize the 2-neutron wave function in a spatial and a spin part: $\psi(x_1, x_2)\,|S, m_S\rangle$ where either $S=0$, $m_S=0$ (antisymmetric) or $S=1$, $m_S = -1, 0,$ or $+1$ (symmetric). Therefore, ψ must be symmetric in x_1, x_2 if $S=0$ and antisymmetric otherwise. Finally, we can factorize $\psi(x_1, x_2)$ as

either $\frac{1}{\sqrt{2}}\left[\psi_m(x_1)\,\psi_n(x_2) + \psi_m(x_2)\,\psi_n(x_1)\right]$ (symmetric)

or $\frac{1}{\sqrt{2}}\left[\quad " \quad\quad " \quad - \quad " \quad\quad " \quad\right]$ (antisymmetric)

IF $m \neq n$; otherwise $\longrightarrow$ $\psi_n(x_1)\,\psi_n(x_2)$ (symmetric only)

The eigenvalues of the 2-neutron Hamiltonian $H = H_1 + H_2$ are the same for the first 2 cases: $E_{n,m} = \frac{\hbar^2}{2mL^2}(n^2 + m^2)$ and $E_{n,n} = \frac{2\hbar^2}{2mL^2} n^2$ otherwise (last case).

Obviously, the lowest possible energy corresponds to $n = m = 1$

$\Rightarrow$ 1 non-degenerate ground state: $\psi_1(x_1)\,\psi_1(x_2)\,|S=0, m_S=0\rangle$ with the eigenvalue (energy) $\frac{2\hbar^2}{2mL^2} = E_{1,1}$.

The next higher energy is $E_{1,2} = \frac{5\hbar^2}{2mL^2} \Rightarrow$

1) cont'd

with 4 eigenstates (4x degenerate):

$$\frac{1}{\sqrt{2}}\left[\psi_1(x_1)\,\psi_2(x_2) + \psi_2(x_1)\,\psi_1(x_2)\right]|S=0, m_s=0\rangle \quad \text{and}$$

$$\frac{1}{\sqrt{2}}\left[\psi_1(x_1)\,\psi_2(x_2) - \psi_2(x_1)\,\psi_1(x_2)\right]|S=1, m_s=\begin{smallmatrix}-1\\0\\+1\end{smallmatrix}\rangle$$

For the next higher energy, we have $E_{2,2} = \frac{\hbar^2}{2mL^2}(4+4)$ which is lower than $E_{1,3} = \frac{\hbar^2}{2mL^2}(1+9)$. There is a single (non-degenerate) state $\psi_2(x_1)\,\psi_2(x_2)\,|S=0, m_s=0\rangle$ for $E_{2,2}$ and 4 states for $E_{1,3}$ (one $S=0$, spatially symmetric, 3 $S=1$, spatially antisymmetric) and so on.

[The next eigenvalues in ascending order are $E_{2,3}, E_{1,4}, E_{3,3}, E_{2,4} \ldots$]

PROBLEM 2

1) Since the Hamiltonian in region II ($x \geq 0$) contains the term $-\gamma \vec{B}\cdot\vec{S} = -\gamma B S_z$, we must find simultaneous eigenfunctions of H and S_z. So for $m_s = +\frac{1}{2}$, the Schrödinger equation reads

$$(*) \quad -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_+^{I}(x) = E\,\psi_+^{I}(x), \quad x<0$$

$$(**) \quad -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_+^{II}(x) - \gamma B\frac{\hbar}{2}\,\psi_+^{II}(x) = E\,\psi_+^{II}(x), \quad x \geq 0$$

Let $k_0 = \frac{p}{\hbar} = \frac{\sqrt{2mE}}{\hbar}$ and $k_1 = \frac{\sqrt{2m(E+\gamma B\hbar/2)}}{\hbar}$

$$\Rightarrow \psi_+^{I}(x) = e^{ik_0x} + A_+e^{-ik_0x}; \quad \psi_+^{II}(x) = B_+e^{ik_1x}$$

2.1 cont'd) $x=0$: $1+A_+ = B_+$; $k_0(1-A_+) = k_1 B_+ = k_1(1+A_+)$

$$\Rightarrow k_0 - k_1 = (k_0+k_1)A_+ \text{ or } A_+ = -\frac{k_1-k_0}{k_1+k_0}, \quad B_+ = \frac{2k_0}{k_1+k_0}$$

2.2:) $$T_+ = \frac{k_1}{k_0}|B_+|^2 = \frac{j(x>0)}{j_{in}} = \frac{4k_0k_1}{(k_1+k_0)^2}$$

$$R_+ = |A_+|^2 = \frac{(k_1-k_0)^2}{(k_1+k_0)^2} \; ; \quad R_+ + T_+ = 1$$

Problem 3

$$P_{0\to n}(\tau) = |d_n(\tau)|^2 \text{ with } d_n(\tau) = \frac{1}{i\hbar}\int_0^\tau e^{i\omega_{fi}t'}\langle n|ex\mathcal{E}_0 e^{-i\omega_p t'}|0\rangle dt'$$

with $\omega_{fi} = \frac{E_n - E_0}{\hbar} = n\omega_0 \Rightarrow d_n(\tau) = \frac{e\mathcal{E}_0}{i\hbar}\int_0^\tau e^{i(n\omega_0-\omega_p)t'}dt' \langle n|x|0\rangle$

Since $x = \sqrt{\frac{\hbar}{2m\omega_0}}(a^\dagger + a)$, $\langle n|x|0\rangle = 0$ for $n>1$

and $= \sqrt{\frac{\hbar}{2m\omega_0}}$ for $n=1$. (We also know that $d_n(\tau)$ becomes large only if $\omega_{fi} \approx \omega_p \Rightarrow n=1$ again.

$$\Rightarrow P_{0\to 1} = \frac{e^2\mathcal{E}_0^2}{\hbar^2}\frac{\hbar}{2m\omega_0}\left|\int_0^\tau e^{i(\omega_0-\omega_p)t'}dt'\right|^2 =$$

$$\frac{e^2\mathcal{E}_0^2}{2\hbar m\omega_0}\left|\frac{e^{i(\omega_0-\omega_p)\tau}-1}{i(\omega_0-\omega_p)}\right|^2 = \frac{e^2\mathcal{E}_0^2\tau^2}{2\hbar\omega_0 m}\left(\frac{\sin[(\omega_0-\omega_p)\tau/2]}{(\omega_0-\omega_p)\tau/2}\right)^2$$

Problem 3 cont'd

For small τ, $P_0 \left| \frac{\sin[(\omega_0-\omega_p)\frac{\tau}{2}]}{(\omega_0-\omega_p)\frac{\tau}{2}} \right| \simeq 1$

$$\Rightarrow \quad P_{0\to 1} \simeq \frac{e^2 \mathcal{E}_0^2 \tau^2}{2\hbar m \omega_0}$$ ("short" means that $\tau(\omega_0-\omega_p) << 1$)

For large τ and $\omega_0-\omega_p \neq 0$, $P_{0\to 1} = \frac{2e^2\mathcal{E}_0^2}{\hbar m \omega_0 (\omega_0-\omega_p)^2} \sin^2(\omega_0-\omega_p)\frac{\tau}{2}$

which oscillates between 0 and a maximum of

$\frac{2e^2\mathcal{E}_0^2}{\hbar m \omega_0 (\omega_0-\omega_p)^2}$ as τ increases. Of course, if $\omega_0=\omega_p$,

τ will never be "long" and the first result obtains for all τ (or, rather, until P is no longer $<<1$ and 1st order PT breaks down).

Problem 4

a) For $B=0$, $H_0 = \frac{\vec{p}^2}{2M} - \frac{e^2}{r}$ with the hydrogen atom

eigenfunctions $|n, l, m\rangle$, $0 \le l \le n-1$, $-l \le m \le l$.

Ignoring $\frac{e^2}{2mc^2}A^2$, the addition due to $B \neq 0$ is

$$H_p = \frac{e}{2mc}(\vec{P}\cdot\vec{A} + \vec{A}\cdot\vec{P}) = \frac{e}{2mc}\frac{\hbar}{i}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\frac{r\sin\theta}{2}B + \frac{r\sin\theta}{2}B\cdot\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

Clearly, $\vec{p}$ commutes with $\vec{A}$ and we get

$$H_p = \frac{e}{2mc}\frac{\hbar}{i}\left(B\frac{\partial}{\partial\varphi}\right) = \frac{e}{2mc}BL_z.$$

Problem 4 cont'd

b) i) $\langle 2,0,0|L_z|2,0,0\rangle = \langle 2,1,0|L_z|2,1,0\rangle = 0$

$\Rightarrow$ there is no first-order shift due to H_p for these EF.

& $\Delta E^{1st\ order}_{\ell m = 1,\pm 1} = \frac{eB}{2mc}\langle 2,1,\pm 1|L_z|2,1\pm 1\rangle = \pm\frac{eB\hbar}{2mc}$

ii) The four $n=2$ EFs above are all degenerate in energy (they form a degenerate subspace). To avoid problems, we need a basis of that subspace which diagonalizes H_p. Fortunately, since the standard hydrogen atom WF's are already eigenstates to L_z (and thus H_p), H_p is already diagonal $\rightarrow$ nothing to do.

iii) $$|\delta\varphi_1\rangle_{\ell_0,m_0} = \sum_{|n,\ell m\rangle \neq |2,\ell_0,m_0\rangle} \frac{\langle n\ell m|H_p|2,\ell_0,m_0\rangle}{E_n - E_2}|n\ell m\rangle$$

However, because all $|n\ell m\rangle$ are EF's to H_p (but $\hat{z}$),
this is always zero $\rightarrow$ there is no 1st order (or ANY order!) change in the EFs

iv) For exactly the same reason, there is no 2nd order change in E – in fact,

$E^B = -\frac{Ry}{4} + \frac{eB\hbar}{2mc}\cdot m$ to all orders for all $|n,\ell,m\rangle$

Problem 5

a) For the numerical values given, $\omega_{fi} = \frac{1}{6.58 \cdot 10^{-16}} Hz = 1.52 \cdot 10^{15} Hz$

This is different from $\omega_p = 10^{-15} Hz$. Because of the δ-function in FGR, $\frac{dP(i \to f)}{dt} = 0$

b) E.g., a state for two spin-$\frac{1}{2}$ particles coupled to total spin $S=0$: $\frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)$

$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right]$

cannot be written as $\begin{pmatrix}a\\b\end{pmatrix}\otimes\begin{pmatrix}c\\d\end{pmatrix}$ for any two 1-particle states $\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix}$

c) Since the dipole moment operator $-e\vec{r}-$ is a rank-1 spherical tensor, the Wigner Eckard theorem says J must be able to couple with 1 to yield j again $\to$ $|j-1| \le j \le j+1$. this is impossible for $j=0$ $(1 \ne 0)$ but possible for $j = \frac{1}{2}$ $(\frac{1}{2} = \frac{1}{2}!) \Rightarrow j = \frac{1}{2}$ is the minimum.

d) According to the WKB method, $\psi(x) = \frac{A}{\sqrt{p(x)}} \cdot e^{i\phi(x)}$

where $\phi(x) = \frac{1}{\hbar}\int_{x_0}^{x} |p(x')|\, dx'$ is some phase.

$dP(x_1 \ldots) = \frac{|A|^2}{p(x_1)} dx_1$ and $dP(x_2 \ldots) = \frac{|A|^2}{p(x_2)} dx$

$\Rightarrow$ Ratio $= \frac{p(x_2)}{p(x_1)} = \frac{\sqrt{2m(E-V(x_2))}}{\sqrt{2m(E-V(x_1))}}$. This also makes sense classically, since $dP \sim \frac{dx}{v} = dT$ (time spent near x_i).

e) According to the variational method, $E_0 \le \min_i \langle i|H|i\rangle$ and the basis state $|j\rangle$ with the lowest $\langle j|H|j\rangle$ has the best chance to be "close to" the ground state.

5f i) $\vec{P} = \frac{1}{4}\hat{x} + \frac{1}{4}\hat{y} + \frac{1}{4}\hat{z}$

$$\rho = \frac{1}{2}(\vec{\sigma}\cdot\vec{P}) + \frac{1}{2}\sigma_0 = \begin{pmatrix} 5/8 & 1/8(1+i) \\ 1/8(1-i) & 3/8 \end{pmatrix}$$

5g): $$\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2} = \frac{0.00997}{(1\ nm^{-1})^2} = 10^{-2}\ nm^2$$

$$= 10^{-16}\ cm^2 \text{ per sr}$$