Quantum Mechanics II Lecture 1

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State Vectors

As we learned last semester, the state of a particle in a system is represented by a state vector $|\psi\rangle$ in some Hilbert space. For each observable associated with the system there is a hermitian operator $\mathcal{O}$ in that same Hilbert space. If this observable is measured it will yield, as a result, one (and only one) of the eigenvalues of $\mathcal{O}$, o_i. In general, even if we know everything there is to know about $|\psi\rangle$, we can only know the *probability* of a measurement yielding a specific eigenvalue, o_i, of $\mathcal{O}$. The exception is, of course, when $|\psi\rangle$ happens to be an eigenstate of $\mathcal{O}$.

Probability

To find the probability, $P(o_i)$, of measuring a certain result, o_i , we will need the projection operator

$$\mathbb{P}_{o_i} = |o_i\rangle \langle o_i| \tag{1}$$

that gives the projection of the state vector onto the eigenstate $|o_i\rangle$. In general, there may be some other quantum number α_j that is needed to completely define the eigenstate (i.e. the state is degenerate). In this case the projection operator will be

$$\mathbb{P}_{o_i} = \sum_j |o_i, \alpha_j\rangle \langle o_i, \alpha_j| \tag{2}$$

Operating on the state vector with the projection operator will give the overlap, or projection, of $|\psi\rangle$ onto the eigenstate, $|o_i\rangle$ (assuming the eigenstate is non-degenerate for simplicity). Operating on this new state vector with the original state vector tells us the probability of $|\psi\rangle$ being in the eigenstate $|o_i\rangle$.

$$\mathbb{P}_{o_i} |\psi\rangle \rightarrow \text{The overlap of } |\psi\rangle \text{ and } |o_i\rangle$$
$$\langle\psi|\mathbb{P}_{o_i}|\psi\rangle \rightarrow \text{The probability, } P(o_i), \text{ of finding } |\psi\rangle \text{ in the state } |o_i\rangle$$

Average Value

One question we can ask ourselves is what would the average be after measuring an operator over and over again? To find the average value (also called the expectation value) of an operator for a given state vector we simply take the sum of each eigenvalue of the operator multiplied with the probability of being in the eigenstate associated with that eigenvalue.

$$\langle \mathcal{O} \rangle_{\psi} = \sum_{i} P(o_i) \cdot o_i \tag{3}$$

It just so happens that the expectation value of an operator is equivalent to operating on the state vector with the operator and then operating on the resultant state vector with the original state vector.
Proof:

$$\begin{aligned}
\langle \mathcal{O} \rangle_{\psi} &= \sum_{i} P(o_i) \cdot o_i \\
&= \sum_{i} \langle \psi | o_i \rangle \langle o_i | \psi \rangle \cdot o_i \\
&= \langle \psi | \sum_{i} o_i \cdot | o_i >< o_i | \, | \psi \rangle \\
&= \langle \psi | \sum_{i} \mathcal{O} \, | o_i >< o_i | \, | \psi \rangle \\
&= \langle \psi | \mathcal{O} | \psi \rangle
\end{aligned} \tag{4}$$

The next obvious question would be how far off this average will usually be from an actual result? This difference (called the standard deviation), can be found by taking the square root of the expectation value of the square of the operator minus the square of the expectation value of the operator.

$$\Delta \mathcal{O}_{\psi} = \sqrt{\langle \mathcal{O}^2 \rangle_{\psi} - \langle \mathcal{O} \rangle_{\psi}^2} \tag{5}$$

For the square of an operator, or, more generally, to the n^{th} power, the definition of the expectation value is only slightly modified:

$$\langle \mathcal{O}^n \rangle_{\psi} = \sum_{i} P(o_i) \cdot o_i^n = \langle \psi | \mathcal{O}^n | \psi \rangle \tag{6}$$

Multi-particle System

Assume that we have N_{part} particles, **all** in the same system $|\psi\rangle$ with **no** interaction between particles.[1] Now, instead of finding the expectation value of a certain operator, we want to know the expectation value of *the number of times a certain eigenvalue is measured*[2], denoted $N(o_i)$.

$$\langle N(o_i)\rangle = N_{part} \cdot P(o_i) \tag{7}$$

Since $N(o_i)$ is a random variable we can calculate its probability:

$$P(n) = \frac{N_{part}!}{n!(N_{part}-n)!} P(o_i)^n (1-P(o_i))^{N_{part}-n} \tag{8}$$

where $N(o_i)$ has been replaced with n to slightly simplify the equation. This will give a distribution of probabilities, the mean of which is given by Eq. 7 (often called μ), and the standard deviation is given by

$$\sigma = \Delta n = \sqrt{< n^2 > - < n >^2} = \sqrt{N_{part} \cdot P(o_i) \cdot (1-P(o_i))} \tag{9}$$

Poisson Distribution

There is an approximation to the exact distribution, called the Poisson Distribution, that only requires that μ is known:

$$P(n) = \frac{\mu^n}{n!} e^{-\mu} \tag{10}$$

There is a catch, however: the distribution is only accurate when the sample size (N_{part}) is very large, $P(o_i)$ is small, and μ is finite.

Gaussian Distribution

Another distribution is known as the Gaussian, which depends on both μ and σ:

$$P(n) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \tag{11}$$

You'll notice that this distribution also depends on x, which is a continuous variable. This means that, in order to be accurate, there will need to be a very large number of "bins", or possible outcomes.

[1] This can be achieved by either having a "bag" of identical particles that can't see each other, or running the same experiment N_{part} times.

[2] For example, how often would we expect tails to turn up exactly 7 times out of 10 fair coin tosses?