Time dependent perturbation theory.

$$H(t) = H_0 + H_p(t) \quad \text{——①}$$

At $t=0$, system is in the eigen state $|n\rangle$ of H_0.

$$H_0|n\rangle = E_n|n\rangle$$

Since the $|n\rangle$ of H_0 forms a complete basis, we can always write;

$$|\psi(t)\rangle = \sum_n a_n(t)\,|n\rangle \quad \text{——②}$$

Here, $\quad a_n(t) = d_n(t)\,e^{-iE_nt/\hbar} \qquad ; \; a_n(0) = \delta_{ni} \quad ; \quad d_n(0) = \delta_{ni}$

Plugging the equation ① in SE;

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = H_0|\psi\rangle + H_p(t)|\psi\rangle$$

②⇒

$$\sum_n i\hbar\left[\frac{\partial d_n(t)}{\partial t}e^{-iE_nt/\hbar} - d_n(t)\frac{iE_n}{\hbar}e^{-iE_nt/\hbar}\right]|n\rangle = \sum_n d_n(t)e^{-iE_nt/\hbar}E_n|n\rangle + \sum_n d_n(t)e^{-iE_nt/\hbar}H_p|n\rangle$$

$$\sum_n \dot{d}_n(t)\,e^{-iE_nt/\hbar}\,|n\rangle = \frac{1}{i\hbar}\sum_n d_n(t)\,e^{-iE_nt/\hbar}\,|n\rangle$$

Multiply this with $\quad \langle f|\exp(iE_f t/\hbar)$

$$\langle f|\sum_n \dot{d}_n(t)\exp\left(\frac{i}{\hbar}(E_f - E_n)t\right)|n\rangle = \frac{1}{i\hbar}\langle f|\sum_n d_n(t)\exp\frac{i}{\hbar}(E_f-E_n)t\cdot H_p|n\rangle$$

$$\dot{d}_f = \frac{1}{i\hbar}\sum_n d_n\exp\frac{i}{\hbar}(E_f-E_n)t\,\langle f|H_p|n\rangle$$

0th approximation;

Assume the sys^{m} is in the same state.

At $t=0$ $\Rightarrow$ system is in the state $|i\rangle$

$$\therefore \; d_n(0) = \delta_{ni}$$
$$\Rightarrow \dot{d}_f = 0$$

1st approximation;

$d_n = 0$ ~~exp~~ except d_i ; $d_i = 1$

$$\therefore \dot{d}_f = \frac{1}{i\hbar} \exp \frac{i(E_f - E_i)t}{\hbar} \langle f | H_p | i \rangle \quad ; \quad f \neq i$$

Integrating $t = 0 \rightarrow t$

$$d_f(t) = \frac{1}{i\hbar} \int_0^t \exp i\omega_{fi} t' \langle f | H_p(t') | i \rangle \, dt' \quad ③ \quad ; \quad \omega_{fi} = \frac{E_f - E_i}{\hbar}$$

Probability $(i \underset{t}{\rightarrow} f) = |d_f(t)|^2 \ll 1$ for $f \neq i$

If $t = -T$ instead $t = 0$;

$$d_f(t) = \frac{1}{i\hbar} \int_{-T}^{t} \exp i\omega_{fi} t' \langle f | H_p(t') | i \rangle \, dt'$$

Consider the harmonic oscillator;

$H_0 = $ H.O.

$t \rightarrow -\infty$; $|0\rangle$

For this system $H_p(t) = e\mathcal{E}x\, e^{-t^2/\tau^2}$ is applied between $t = -\infty \longrightarrow \infty$

Here $x = \sqrt{\frac{\hbar}{2m\omega}} \, (a^{+} + a)$

$$\langle f | H_p(t') | i \rangle = \langle f | e\mathcal{E}x\, e^{-t^2/\tau^2} | 0 \rangle$$

Only non zero term is , $\langle 1 | e\mathcal{E} e^{-t^2/\tau^2} x | 0 \rangle$

From eq^n ③ $\Rightarrow$ $d_1(t) = \frac{1}{i\hbar} \int_{-\infty}^{\infty t} e^{i\omega t'} \cdot e^{-t^2/\tau^2} \sqrt{\frac{\hbar}{2m\omega}}\, dt'$; $\omega = \omega_{fi}$

What happens if $t \to \infty$?

$$d_1(\infty) = \frac{1}{i\hbar} \int_{-\infty}^{\infty} \exp\left(i\omega t' - \frac{t'^2}{\tau^2} \right) \sqrt{\frac{\hbar}{2m\omega}}\, dt'$$

$$d_1(\infty) = \frac{1}{\sqrt{2m\omega\hbar}} \frac{e\mathcal{E}}{i} \sqrt{\pi}\, \tau \exp\left(-\frac{\omega^2\tau^2}{4} \right)$$

$\therefore$ Prob. of transition $0 \to 1$ as $t \to \infty$ $= \frac{\pi e^2 \mathcal{E}^2 \tau^2}{2m\omega\hbar} \exp\left(-\frac{\omega^2\tau^2}{2} \right)$

If $\tau \lll$ it doesn't have a time to do anything ; remains in the same state.

$\therefore$ $P \sim 0$

If $\tau \ggg$ $P \sim 0$

The sudden perturbation:

Here the Hamiltonian changes over a small interval.

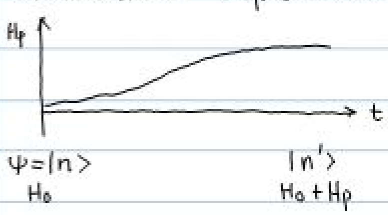
$\tau \to 0$ $\quad\quad d_{fi} = \delta_{fi}$ $\quad\quad \Rightarrow P \sim 0$

Consider a particle in a box (HW SET 05 - problem 2)
If the walls moves out suddenly?

- First, nothing happens. Wave function will be same.
- Over time it changes.

Consider a tritium atom (have same hydrogen wf)
Here is e bound to a nucleus of charge Z undergoes β 'ay by emitting a relativistic e and changing its charge to $Z+1$
Since this time is very small, the state of the atomic e is the same just before & just after β decay.

Adiabatic perturbation:



$\tau \to$ large ; $P \sim 0$

Here the hamiltonian changes slowly. Therefore the system can adjust to those changes.

$$1/\tau \lll \min_f \left(\frac{E_f - E_i}{\hbar} \right) \quad \text{or} \quad \tau \ggg 1/\omega_{min}$$

Periodic Perturbation

$$H_p(t) = h_p \exp(-i\omega_p t) \quad \theta(t) \quad ; \quad t > 0$$

The amplitude for the transition from $|i\rangle \to |f\rangle$ in time t ;

$$d_f(t) = \frac{1}{i\hbar} \int_0^t \exp i(\omega_{fi} - \omega_p)t' \quad \langle f| h_p |i\rangle \quad dt'$$

$$= \langle f | h_p | i\rangle \quad \frac{1}{i\hbar} \quad \frac{\exp i(\omega_{fi} - \omega_p) - 1}{i(\omega_{fi} - \omega_p)}$$

$$= i\, t \quad \exp i(\omega_{fi} - \omega_p)t/2 \quad \frac{\sin(\omega_{fi} - \omega_p)t/2}{(\omega_{fi} - \omega_p)t/2 \cdot \hbar} \cdot \langle f|h_p|i\rangle$$

$$\text{Probability } (i \to f) = |\langle f|h_p|i\rangle|^2 \; \frac{t^2}{\hbar^2} \; \underbrace{\frac{\sin^2(\omega_{fi} - \omega_p)t/2}{[(\omega_{fi} - \omega_p)t/2]^2}}_{x(t)}$$

$(\omega_{fi} - \omega_p)t = 2\pi$

For $t \to \infty$;

$$d_f(\infty) = \frac{1}{i\hbar} \int_{-\infty}^{\infty} \exp i(\omega_{fi} - \omega_p)t' \quad \langle f|h_p|i\rangle \quad dt'$$

$$= \frac{\langle f|h_p|i\rangle}{i\hbar} \cdot 2\pi \; \delta(\omega_{fi} - \omega_p)$$

$$\text{Prob}\ (i \to f) = |d_f|^2$$

$$= \frac{|\langle f | \cancel{H}\, h_p | i \rangle|^2}{\hbar^2} (2\pi)^2 \, \delta(\omega_{fi} - \omega_p) \cdot \delta(\omega_{fi} - \omega_p)$$

$$\delta \cdot \delta = \lim_{T \to \infty} \delta(\omega_{fi} - \omega_p) \frac{1}{2\pi} \int_{-T/2}^{T/2} \exp\, i(\omega_{fi} - \omega_p)t \, dt$$

$$\therefore \quad |d_f|^2 = |\langle f | h_p | i \rangle|^2 \frac{2\pi}{\hbar^2} \delta(\omega_{fi} - \omega_p)\, T$$

$$\text{Average transition rate} = \frac{P_{(i \to f)}}{T}$$

$$\frac{dP}{dt} = |\langle f | h_p | i \rangle|^2 \frac{2\pi}{\hbar^2} \delta(\omega_{fi} - \omega_p)$$

$$= \frac{2\pi}{\hbar} |\langle f | h_p | i \rangle|^2 \, \delta(E_f - E_i - \hbar\omega_p)$$

— Fermi's Golden Rule —