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Higher orders in PT

Schrödinger Picture.

Here the state of the particle is described by a vector $|\psi_s(t)\rangle$
This state evolves in time but the operators are constant.

Probability for obtaining the result when Ω is measured;

$$P(\omega, t) = |\langle \omega_s | \psi_s(t) \rangle|^2$$

The time evolution of $|\psi_s(t)\rangle$ is given by;

$$i\hbar \frac{d}{dt} |\psi_s(t)\rangle = H_s |\psi_s(t)\rangle \qquad \text{①}$$

$$H_s = H_s^0 + H_s^1(t)$$

$$\therefore \quad i\hbar \frac{d}{dt} |\psi_s(t)\rangle = \left(H_s^0 + H_s^1(t) \right) |\psi_s(t)\rangle$$

If we define a propagator $U_s(t, t_0)$ by;

$$|\psi_s(t)\rangle = U_s(t, t_0) |\psi_s(t_0)\rangle$$

From ① ; $i\hbar \frac{d}{dt} U_s(t,t_0)|\ \rangle = H_s U_s(t, t_0) |\psi_s(t_0)\rangle$

$$i\hbar \frac{dU_s}{dt} = H_s U_s$$; this follows eq^n ①.

The propagator U;

① $U^\dagger U = \mathbb{1}$
② $U(t_3, t_2) U(t_2, t_1) = U(t_3, t_1)$
③ $U(t_1, t_1) = \mathbb{1}$
④ $U^\dagger(t_1, t_2) = U(t_2, t_1)$

Interaction Picture ;

Here the both state vector & the operators carry the time dependance.

$$H = H_0 + H_S^1(t)$$

Suppose our $H = H_S^0$

Then,

$$i\hbar \frac{dU_S^0}{dt} = H_S^0 U_S^0 \quad \text{——— ②} \qquad \Rightarrow U_S^0(t,t_0) = \exp -iH_S^0(t-t_0)/\hbar$$

The state vector in this picture, $|\psi_I(t)\rangle = [U_S^0(t,t_0)]^\dagger |\psi_S(t)\rangle$

$$i\hbar \frac{d}{dt}|\psi_I(t)\rangle = i\hbar \frac{dU_S^{0\dagger}}{dt}|\psi_S(t)\rangle + U_S^{0\dagger} i\hbar \frac{d}{dt}|\psi_S\rangle$$

$$= -U_S^{0\dagger} H_S^0 |\psi_S\rangle + U_S^{0\dagger}(H_0 + H_0^1)|\psi_S\rangle$$

$$= U_S^{0\dagger} H_S^1 |\psi_S\rangle$$

$$= U_S^{0\dagger} H_S^1 U_S^0 U_S^{0\dagger} |\psi_S\rangle$$

$$= H_I^1(t) U_S^0 |\psi_I\rangle$$

$$i\hbar \frac{d}{dt}|\psi_I(t)\rangle = H_I^1(t)|\psi_I(t)\rangle \quad \text{——— ①}$$

$$\hbar \langle \omega_S | \psi_S(t)\rangle = \langle \omega_S | U_S^0(t,t_0) U_S^0(t,t_0)^\dagger | \psi_S(t)\rangle$$

$$= \langle \omega_S | U_S^0(t,t_0) | \psi_I(t)\rangle$$

$$= \langle \omega_I(t) | \psi_I(t)\rangle$$

Time dependent $|\omega_I(t)\rangle$ is the eigen vector of the time dependent operator, $\Omega_I(t) = U_S^{0\dagger} \Omega_S U_S^0$ ——— ③

$$\Omega_I |\omega_I(t)\rangle = U_S^{0\dagger} \Omega_S U_S^0 U_S^\dagger |\omega_S(t)\rangle$$
$$= U_S^{0\dagger} \Omega_S |\omega_S(t)\rangle$$
$$= \omega U_S^{0\dagger} |\omega_S(t)\rangle$$
$$= \omega |\omega_I(t)\rangle$$

Combining ② & ③ ;

$$i\hbar \frac{d\Omega_I(t)}{dt} = i\hbar \frac{dU_S^{0\dagger}}{dt} \Omega_S U_S^0 + U_S^{0\dagger} \Omega_S \, i\hbar \frac{dU_S^0}{dt}$$
$$= U_S^{0\dagger} [\Omega_S, H_S^0] U_S^0$$
$$= [\Omega_I, H_I^0(t)]$$

Define a propagator $U_I(t,t_0)$ in the Interaction picture ;

$$|\Psi_I(t)\rangle = U_I(t,t_0) |\Psi_I(t_0)\rangle$$

From Ⓐ ;

$$i\hbar \frac{dU_I}{dt} = H_I^1 U_I \quad \text{———⑤}$$; follows eq$^{\underline{n}}$ ①

$U_I(t,t_0) |\Psi_I(t)\rangle$ & $U_S^0(t,t_0)^\dagger U_S(t,t_0) |\Psi_S(t)\rangle$

$$U_I(t,t_0) = U_S^0(t,t_0)^\dagger U_S(t,t_0)$$; Once we find $U_I(t)$ we can always go back to $U_S(t)$.

$$U_S^0(t,t_0) U_I(t,t_0) = U_S(t,t_0)$$

Eq^{n} ⑤ solution ;

$$U_I(t,t_0) = C - \frac{i}{\hbar} \int_{t_0}^{t} dt' H_I^1(t') U_I(t',t_0)$$
$$U_I(t_0,t_0) = C = 1$$
$$U_I(t,t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} dt' \, H_I^1(t') U_I(t',t_0) \quad \text{———⑥}$$

To find U_I to zeroth order,

Set $H = H_0 + (H')^0 = H_0$

eq^n ⑥ $\Rightarrow$ $U_I(t,t_0) = \mathbb{1} + O(H_I')$

To find to first order we can keep only H_I' powers only;
So;

$$U_I(t,t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} H_I'(t')\, dt' + O(H_I^2)$$

Transition amplitude; $d_f(t) = \langle f_s^0 |\, \exp iE_f^0(t-t_0)/\hbar \; U_S(t-t_0)\, | i_s^0 \rangle$

$$d_f(t) = \langle f_s^0 |\, \underbrace{\exp iH_S^0(t-t_0)/\hbar}_{U_S^{\dagger}(t,t_0)} \; U_S(t-t_0)\, | i_s^0 \rangle \quad ; \text{ since } H_S^0|i_s^0\rangle = E_i^0|i$$

$$= \langle f_s^0 | \underbrace{U_S^{0\dagger} U_S}_{U_I} | i_s^0 \rangle$$

$$d_f(t) = \langle f_s^0 | U_I | i_s^0 \rangle$$

$$= \underbrace{\langle f_s^0 | i_s^0 \rangle}_{\delta_{fi}} - i/\hbar \int_{t_0}^{t} dt' \; \langle f_s^0 | H_I'(t) | i_s^0 \rangle$$

$$= \delta_{fi} - i/\hbar \int_{t_0}^{t} dt' \; \langle f_s^0 |\, \exp iE_f^0(t'-t)/\hbar \; H_S'(t) \; \exp -iE_i^0(t'-t)/\hbar \, | i_s^0$$

$$= \delta_{fi} - i/\hbar \int_{t_0}^{t} dt' \; \exp i\omega_{fi}(t'-t_0) \; \langle f_s^0 | H_S'(t) | i_s^0 \rangle$$

For Higher Orders ;

$$U_I(t,t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} H_I'(t')\, U_I(t',t_0)\, dt'$$

$$U_I(t,t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} H_I'(t')\left(1 - \frac{i}{\hbar}\int_{t_0}^{t'} H_I'(t'')\, dt''\right)$$

$$= \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, H_I'(t') + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t'} dt'\, H_I'(t') \int_{t_0}^{t'} dt''\, H_I'(t'')$$

$$U_S(t,t_0) = U_S^0(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, U_S^0(t,t_0)\, U_S^{0\dagger}(t',t_0)\, H_S'\, U_S^0(t',t_0)\, dt'$$

$$+ \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, U_S^0(t,t_0)\, U_S^{0\dagger}(t',t_0)\, H_S'\, U_S^0(t',t_0) \times$$

$$\times\, U_S^{0\dagger}(t'',t_0)\, H_S'\, U_S^0(t'',t_0)\, dt'\, dt'' + \ldots$$

$$U_S^0(t,t_0)\, U_S^{0\dagger}(t',t_0) = U_S^0(t,t')$$

$\therefore$

$$U_S(t,t_0) = U_S^0(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, U_S^0(t,t')\, H_S'\, U_S^0(t',t_0)\, dt'$$

$$+ \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, U_S^0(t,t')\, H_S'\, U_S^0(t',t'')\, H_S'\, U_S^0(t'',t_0).$$

- On the LHS we have complete Schrödinger picture propagat
- First term of RHS says system evolves from $t_0 \to t$ in response to U_S^0
- Second term ;
  System evolves from $t_0 \to t'$ in response to U_S^0.
  There it interact with perturbation & thereafter response to U_S^0 alone until time t.