Graduate Quantum Mechanics – Final Exam - Solution

Problem 1)

The eigenstates of the "particle in a box" problem have eigenvalues of $E_n = n^2 \frac{\hbar^2\pi^2}{2mL^2}$. Like all bound states, the expectation value for the momentum operator is zero. This can also be seen from the fact that the eigenfunctions are sinusoidal and applying the momentum operator will convert the sin to a cos – which then has to be multiplied with the sin and integrated over dx (over the interval L where the sin's and cos's are orthogonal to each other). Furthermore, we observe that $\mathbf{P}^2 = 2m\mathbf{H}$ so the expectation value for is simply $\langle \mathbf{P}^2 \rangle = n^2 \frac{\hbar^2\pi^2}{L^2} \Rightarrow \quad \Delta P = \sqrt{\langle \mathbf{P}^2 \rangle - \langle \mathbf{P} \rangle^2} = n\frac{\hbar\pi}{L}$. Multiplying that with the given $\Delta x = L/2$ we see that the product $\Delta x\, \Delta p = n\hbar\pi/2$ which is definitely larger than the minimum allowed by the Heisenberg Uncertainty relationship.

Problem 2)

a) Multiplying out the given expression for the Hamiltonian, we find

$$\mathbf{H} = \frac{\left(\mathbf{P}_x + \mathbf{Y}\frac{qB}{2}\right)^2 + \left(\mathbf{P}_y - \mathbf{X}\frac{qB}{2}\right)^2}{2m} = \frac{\mathbf{P}_x^2 + \mathbf{P}_y^2 + (\mathbf{X}^2 + \mathbf{Y}^2)\frac{q^2B^2}{4} + 2\left(\mathbf{P}_x\mathbf{Y} - \mathbf{P}_y\mathbf{X}\right)\frac{qB}{2}}{2m} =$$

$$\frac{\mathbf{P}_x^2 + \mathbf{P}_y^2 + (\mathbf{X}^2 + \mathbf{Y}^2)\frac{q^2B^2}{4} - qB\mathbf{L}_z}{2m}$$

The first two terms are clearly invariant under rotations (and therefore should commute with $\mathbf{L}_z$), and the last term is proportional to $\mathbf{L}_z$ and therefore commutes with it, as well.

b) $\left[\mathbf{V}_x, \mathbf{V}_y\right] = \frac{1}{m^2}\left(\left[\mathbf{P}_x, \mathbf{P}_y\right] - \frac{qB}{2}\left[\mathbf{P}_x, \mathbf{X}\right] + \frac{qB}{2}\left[\mathbf{Y}, \mathbf{P}_y\right] - \frac{q^2B^2}{4}\left[\mathbf{Y}, \mathbf{X}\right]\right) = \frac{qB}{m^2} i\hbar\mathbf{1} = i\frac{\hbar\omega}{m}\mathbf{1}$

c) $\left[\mathbf{A}, \mathbf{A}^\dagger\right] = \frac{m}{2\hbar\omega}\left[\mathbf{V}_x + i\mathbf{V}_y, \mathbf{V}_x - i\mathbf{V}_y\right] = \frac{m}{2\hbar\omega}\left(-i\left[\mathbf{V}_x, \mathbf{V}_y\right] + i\left[\mathbf{V}_y, \mathbf{V}_x\right]\right) = \frac{m}{\hbar\omega}\frac{\hbar\omega}{m}\mathbf{1} = \mathbf{1}$

Similarly,

$$\mathbf{H} = \frac{(m\mathbf{V}_x)^2 + (m\mathbf{V}_y)^2}{2m} = \frac{m}{2}\left(\mathbf{V}_x^2 + \mathbf{V}_y^2\right) = \frac{m}{2}\left(\mathbf{V}_x - i\mathbf{V}_y\right)\left(\mathbf{V}_x + i\mathbf{V}_y\right) - \frac{m}{2}i\left[\mathbf{V}_x, \mathbf{V}_y\right] =$$

$$\frac{m}{2}\frac{2\hbar\omega}{m}\mathbf{A}^\dagger\mathbf{A} - \frac{m}{2}i\left(i\frac{\hbar\omega}{m}\mathbf{1}\right) = \hbar\omega\left(\mathbf{A}^\dagger\mathbf{A} + \tfrac{1}{2}\mathbf{1}\right).$$

d) The commutator between $\mathbf{A}$ and $\mathbf{A}^\dagger$ as well as the form of the Hamiltonian are exactly the same as for the ladder operators and the Hamiltonian for the 1-dimensional harmonic oscillator. Therefore, we expect to find the same eigenvalues. (However, the eigenvectors will be degenerate in general).

Problem 3)

Let's try $\left(\mathbf{A}-\lambda\mathbf{B}\right)^{-1}=\mathbf{A}^{-1}+\lambda\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$. Sure enough, if we multiply with ($\mathbf{A}$ - λ $\mathbf{B}$) on either side and discard all terms proportional to λ^2**,** we find the unit operator.

Problem 4)

a) Since ℓ=1 implies three different possible values for m (+1, 0 ,-1), there are three possible trajectories (corresponding to these three magnetic quantum numbers) that the outgoing atoms can take.

b) The top-most trajectory corresponds to m = 1 (because of the negative value of the electron charge, this follows from the magnetic moment pointing down). Plugging in the numbers in the expression for the Hamiltonian, H = - $\gamma\mathbf{L}_z$B, yields $3\cdot 10^{-4}$ eV. This is about 0.01% of the binding energy, Ry/n^2 = 3.4 eV.

c) The expectation value for the energy is -16/25·Ry - 9/25·Ry/4 = -0.73 Ry. Similarly, the expectation value for $\mathbf{L}^2$ is 16/25·0 + 9/25·2$\hbar^2$ and for $\mathbf{L}_z$ it is 9/25·$\hbar$. None of these expectation values change with time since all three operators commute with the Hamiltonian.

d) Since the Rydberg constant is proportional to Z^2, all energy levels for the new atom are 4 times their value for tritium. To keep the same energy, the electron needs to be in an n=2 (l=0) state.

Problem 5) – *Extra Credit*

a) We know that $\left[\mathbf{L}_x,\mathbf{L}_y\right]=i\hbar\mathbf{L}_z$ and therefore

$$\Delta\mathbf{L}_x\Delta\mathbf{L}_y\geq\frac{1}{2}\left|\left\langle\ell,m\right|\left[\mathbf{L}_x,\mathbf{L}_y\right]\left|\ell,m\right\rangle\right|=\frac{1}{2}\left|\left\langle\ell,m\right|i\hbar\mathbf{L}_z\left|\ell,m\right\rangle\right|=\frac{\hbar^2|m|}{2}$$ according to Heisenberg.

On the other hand, we know that $\left(\Delta\mathbf{L}_x\right)^2=\left\langle\mathbf{L}_x^2\right\rangle-\left\langle\mathbf{L}_x\right\rangle^2=\left\langle\mathbf{L}_x^2\right\rangle$ (and similar for $\Delta\mathbf{L}_y$) so we can write

$$\hbar^2\ell(\ell+1)=\left\langle\mathbf{L}^2\right\rangle=\left\langle\mathbf{L}_x^2\right\rangle+\left\langle\mathbf{L}_y^2\right\rangle+\left\langle\mathbf{L}_z^2\right\rangle=2\left(\Delta\mathbf{L}_x\right)^2+\left\langle\ell,m\right|\mathbf{L}_z^2\left|\ell,m\right\rangle=$$

$$=2\left(\Delta\mathbf{L}_x\right)^2+\left|\mathbf{L}_z\left|\ell,m\right\rangle\right|^2=2\left(\Delta\mathbf{L}_x\right)^2+\left(m\hbar\right)^2\ (\mathbf{L}_z\text{ is hermitian})\Rightarrow$$

$$\Delta\mathbf{L}_x=\Delta\mathbf{L}_y=\sqrt{\hbar^2\frac{\ell(\ell+1)-m^2}{2}}\text{ and }\Delta\mathbf{L}_x\Delta\mathbf{L}_y=\hbar^2\frac{\ell(\ell+1)-m^2}{2}$$

which is clearly larger than required by Heisenberg (m^2 is bounded by ℓ^2). The lower limit and the exact answer are only equal if $m=\ell$.

b)

The Schrödinger equation for this situation reads as follows:

$$\frac{d^2U_{E1}}{dr^2}+\left[k^2-\frac{-2\mu\hbar^2}{\hbar^2\mu r^2}-\frac{1(1+1)}{r^2}\right]U_{E1}=\frac{d^2U_{E1}}{dr^2}+k^2U_{E1}=0 \text{ with } k=\frac{\sqrt{2\mu E}}{\hbar} \text{ for } r\le a \text{ and } U_{E1}=0$$

for $r > a$. The solution (with the requirement that $U_{E0} \to 0$ as $r \to 0$) reads

$$U_{E0}(r)=\begin{cases} A\sin kr,\ r\le a \\ 0,\ r>a \end{cases}$$

The wave function must be continuous at $r = a$, which yields $A\sin ka=0$ and therefore $k=\frac{n\pi}{a}$

or $E_n=\frac{n^2\pi^2\hbar^2}{2\mu a^2}$

Table of Clebsch-Gordan coefficients:

The nomenclature is $\langle j_1, m_1, j_2, m_2 | J, M \rangle$:

$$\left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \middle| 1,1 \right\rangle = 1$$

$$\left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2} \middle| 1,0 \right\rangle = \left\langle \frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2} \middle| 1,0 \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2} \middle| 0,0 \right\rangle = -\left\langle \frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2} \middle| 0,0 \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2} \middle| 1,-1 \right\rangle = 1$$

$$\left\langle 1,1,\frac{1}{2},\frac{1}{2} \middle| \frac{3}{2},\frac{3}{2} \right\rangle = 1$$

$$\left\langle 1,1,\frac{1}{2},-\frac{1}{2} \middle| \frac{3}{2},\frac{1}{2} \right\rangle = \sqrt{\frac{1}{3}}\,; \quad \left\langle 1,0,\frac{1}{2},\frac{1}{2} \middle| \frac{3}{2},\frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}}$$

$$\left\langle 1,1,\frac{1}{2},-\frac{1}{2} \middle| \frac{1}{2},\frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}}\,; \quad \left\langle 1,0,\frac{1}{2},\frac{1}{2} \middle| \frac{1}{2},\frac{1}{2} \right\rangle = -\sqrt{\frac{1}{3}}$$

$$\left\langle 1,0,\frac{1}{2},-\frac{1}{2} \middle| \frac{3}{2},-\frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}}\,; \quad \left\langle 1,-1,\frac{1}{2},\frac{1}{2} \middle| \frac{3}{2},-\frac{1}{2} \right\rangle = \sqrt{\frac{1}{3}}$$

$$\left\langle 1,0,\frac{1}{2},-\frac{1}{2} \middle| \frac{1}{2},-\frac{1}{2} \right\rangle = \sqrt{\frac{1}{3}}\,; \quad \left\langle 1,-1,\frac{1}{2},\frac{1}{2} \middle| \frac{1}{2},-\frac{1}{2} \right\rangle = -\sqrt{\frac{2}{3}}$$

$$\left\langle 1,-1,\frac{1}{2},-\frac{1}{2} \middle| \frac{3}{2},-\frac{3}{2} \right\rangle = 1$$

(all others not listed are zero)