summary- catch up

The Classical Limit

For any conserved quantity → continuity equation
Example: Electric charge -> Divergence of current = time derivative of charge density

In the case of QM, we have the probability density of finding a particle near x; total probability is conserved.
$\rho = |\Psi|^2_{(x)} = \Psi^*_{(x)}\Psi_{(x)} = \langle\Psi|x\rangle\langle x|\Psi\rangle$
probability = $\rho(x)\cdot\Delta x$ ↳ changes with time

$\therefore \rho(x) = \dfrac{\text{probability}}{\Delta x}$ a probability density (in one dimension)

we know that the probability over all x is 1 $\displaystyle\int_{-\infty}^{\infty} \text{probability } dx = 1$

What is the time dependence of the probability density $\dfrac{d\rho_{(x)}}{dt}$

note: probability cannot be created nor destroyed, only moved

this introduces the concept of a probability current, $j_{(x)}$

from $\dfrac{d\rho_{(x)}}{dt} + \dfrac{dj_{(x)}}{dx} = 0$

and assuming a simplified Hamiltonian, $H = \dfrac{P^2}{2m} + V$

Where $V = \displaystyle\int_{-\infty}^{\infty} dx\, |x\rangle V_{(x)} \langle x|$

$V_{(x)}$ = a real function of x

To find $\dfrac{d\rho_{(x)}}{dt} = \dfrac{d}{dt} \langle\Psi|\underbrace{x\rangle\langle x}_{\text{projection operator onto eigenfunction of x, } P_x}|\Psi\rangle$

$\dfrac{(i\hbar)d}{dt}\langle IP_{(x)}\rangle$ ↳ Postulate no. 4

$\dfrac{(i\hbar)d}{dt}\langle IP_{(x)}\rangle = \langle\Psi|[IP_x,H]|\Psi\rangle$

$\displaystyle\int_{-\infty}^{\infty} dx'|x'\rangle V_{(x')}\langle x'|$

$[IP_x,H] = [IP_x,P^2/2m] + [IP_x,V]$

$[IP_x,V] = 0 = [|x\rangle\langle x|,V] = 0 = |x\rangle V_{(x)}\langle x| - |x\rangle V_{(x)}\langle x|$ ($\neq$ true if $V \neq V_{(x)}$)

summary- catch up

singular, but complex conjugate

$$\therefore [IP_x, H] = [IP_x, P/2m] = \underbrace{\langle \Psi \| x \rangle}_{\Psi^*_{(x)}} \langle x | P^2/2m | \Psi \rangle - \langle \Psi | P^2/2m | x \rangle \underbrace{\langle x \| \Psi \rangle}_{\Psi_{(x)}} \qquad \frac{-\hbar^2}{2m} + \frac{\partial^2 \Psi^*_{(x)}}{\partial x^2}$$

figure out by writing in full coordinate representation (i.e. in terms of x)

$\hookrightarrow \frac{-\hbar^2}{2m} + \frac{\partial^2 \Psi_{(x)}}{\partial x^2}$

$$= \Psi^*_{(x)} \left(\frac{-\hbar^2}{2m} + \frac{\partial^2 \Psi_{(x)}}{\partial x^2} \right) - \left(\frac{-\hbar^2}{2m} + \frac{\partial^2 \Psi^*_{(x)}}{\partial x^2} \right) \Psi_{(x)}$$

By factoring out a negative $\frac{\partial}{\partial x}$

$$= \frac{-\partial}{\partial x} \left[\frac{\hbar^2}{2m} \left(\Psi^*_{(x)} \frac{\partial \Psi_{(x)}}{\partial x} - \frac{\partial \Psi^*_{(x)}}{\partial x} \Psi_{(x)} \right) \right]$$

Now factoring out a constant, i

$$= \frac{-\partial}{\partial x} \left[\underbrace{\frac{\hbar^2}{2mi} \left(\Psi^*_{(x)} \frac{\partial \Psi_{(x)}}{\partial x} - \frac{\partial \Psi^*_{(x)}}{\partial x} \Psi_{(x)} \right)}_{= j_{(x)}} \right]$$

We can also find $j_{(x)}$ in term of momentum

$$\langle \Psi \| x \rangle \langle x | P^2/2m | \Psi \rangle - \underbrace{\langle \Psi | P^2/2m | x \rangle \langle x \| \Psi \rangle}_{\text{exact imaginary part}}$$

classically: $j_{(x)} = pV_{local}$

Commentary, in 3D j becomes $\vec{j}$

$$\vec{j} = \frac{\hbar^2}{2mi} \left[\Psi^* \vec{\nabla} \Psi - (\vec{\nabla} \Psi^*) \Psi \right]$$

$$\frac{d\rho}{dt} + \vec{\nabla} \vec{j} = 0$$

summary- catch up

Classical Limit

explain commutators and translation of variables

We know, for any variable $(i\hbar)\frac{d}{dt}\langle O\rangle_\psi = \langle [O,H]\rangle_\psi$

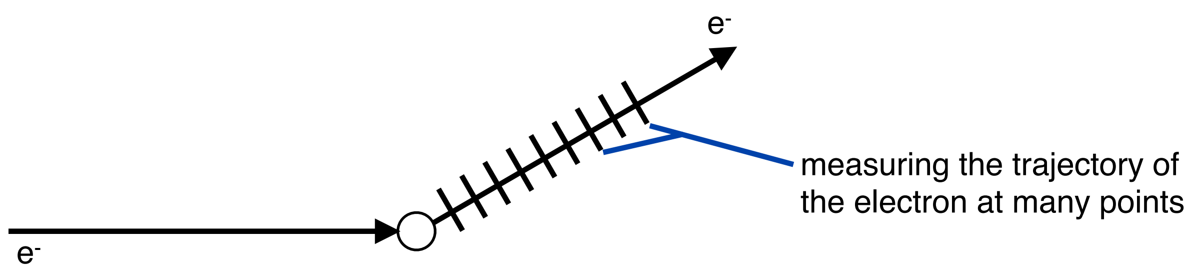
If quantum mechanics is the true theory of the universe then clearly classical mechanics is wrong, but conversely classical mechanics is extremely successful.

To ratify this we ask how is quantum mechanics connected to classical mechanics?

We found before, a gaussian packet is a good representation of a particle.

The ideal particle, i.e. one in which there is no uncertainty, is an ideal construction.

e.g. an e^{-1} that is scattered at J-Lab, with 1 GeV = energy, $\therefore$ momentum, P = 1Gev/c



If we were to turn on a magnetic field we would see a trajectory closer to,

With large detectors, “quantum uncertainty” can be neglected, a typical good measurement is about 100µm, for the purpose of this discussion let these measurements be as low as 100 nm or even 1 nm (even though this is technically impossible)

$\Rightarrow$ resolution in position, σ = 1nm

$$\sigma_p = \frac{\hbar}{2\sigma} = 100\ \frac{eV}{c}$$

Typical momenta are 10^9 eV/c or more so this is a "perturbation" of only 10^{-7}. Even if we measure position 1 million times, we still get only an uncertainty of 10^{-4}.

∴ a single e^- can be talked about in a classical track.

What about it's spread?

if we know momentum

$$t: \sigma = (1\text{nm})^2 + \underbrace{\frac{\sigma_p^2 t^2}{m^2}}_{(2\cdot 10^{-4}c)^2 t^2}$$

$m_e = 511{,}000$ eV/c^2

$$c = 3\cdot 10^{17}\ \text{nm/s} \Rightarrow 2\cdot 10^{-4}c = 6\cdot 10^{13}\text{nm/s}$$

↳ uncertainty "spread"

∴ after 10^{-14} s, spread noticeably increases. However, if the measurement accuracy would have been 1 μm = 1000 nm, it would take 10 ns (the length of the whole trajectory in a typical detector) for the spreading to make the position noticeably more uncertain.
So why doesn't the electron continue to spread?

→Every position measurement resets the clock on the e^- and stops it spreading out to the entire universe.

we can similarly say for everyday objects;

- hard to measure accurately enough to appreciate any spread

- they are continually measured.

summary- catch up

Formally

Classical mechanics

$$\boxed{\begin{aligned}\frac{dx}{dt} &= \frac{\partial H}{\partial p}\\ \frac{dp}{dt} &= \frac{-\partial H}{\partial x}\end{aligned}}$$

What happens to the averages in QM?
<x>, <p> → knowing these values will give you the most exact classical mechanics answer.

↱ where X is the position operator

$$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\langle\Psi|\,[X,H]\,|\Psi\rangle$$

assume $H = \dfrac{P^2}{2m} + V$

$$= \frac{1}{i\hbar 2m}\langle\Psi|\, X(P^2)-(P^2)X\,|\Psi\rangle$$

using the commutator relation $[\Omega,\Lambda^2] = \Omega\Lambda^2-\Lambda^2\Omega + \overbrace{\Lambda\Omega\Lambda-\Lambda\Omega\Lambda}^{\text{adding zero}}$
$= [\Omega,\Lambda]\Lambda-\Lambda[\Lambda,\Omega]$
or $[\Omega,\Lambda]\Lambda+\Lambda[\Omega,\Lambda]$

$\therefore [X,P^2] = [X,P]P + P[X,P] = 2\hbar iP$

↳ $[X,P] = i\hbar$, very, very important

$$\therefore \quad \frac{1}{i\hbar 2m}\langle\Psi|\, X(P^2)-(P^2)X\,|\Psi\rangle = \frac{1}{m}\langle\Psi|\, P\,|\Psi\rangle = \frac{\langle P\rangle}{m}$$

which agrees with the classical work for this Hamiltonian

What we really want is $\dfrac{1}{i\hbar 2m}\langle\Psi|\, X(P^2)-(P^2)X\,|\Psi\rangle = \left\langle\dfrac{\partial H}{\partial P}\right\rangle$

this is resolved with the correct interpretation

↳ might not be true for other Hamiltonians

for the particle in a box problem $\dfrac{\partial\langle x\rangle}{\partial t} = 0$ (true but not useful)

useful for particle with energy > max V at ∞

any bound state must have zero average momentum

summary- catch up

Now examine $\frac{\partial \langle P \rangle}{\partial t}$

$$\frac{\partial \langle P \rangle}{\partial t} = \frac{1}{i\hbar} \langle \Psi | \underbrace{[P,H]}_{} | \Psi \rangle$$

this time $[P, P^2/2m] = 0$

$$= \frac{1}{i\hbar} \langle \Psi | [P,V] | \Psi \rangle$$

$$= \int dx \, \frac{1}{i\hbar} \left(\Psi^*_{(x)} \frac{\hbar}{i} \frac{\partial V_{(x)}}{\partial x} \cdot \Psi_{(x)} - \Psi^*_{(x)} \frac{\hbar}{i} V_{(x)} \frac{\partial \Psi_{(x)}}{\partial x} \right)$$

$$= - \Psi^*_{(x)} \frac{\partial V_{(x)}}{\partial x} \cdot \Psi_{(x)} = \langle \Psi | - \frac{\partial V_{(x)}}{\partial x} | \Psi \rangle$$

$$= \left\langle - \frac{\partial V}{\partial x} \right\rangle \qquad = \left\langle - \frac{\partial H}{\partial x} \right\rangle$$

note:we retrieve something in quantum mechanics that agrees with classical mechanics

However, for perfect agreement we would require that this is $= - \frac{\partial H}{\partial \langle x \rangle}$

There might be a difference between $\left\langle - \frac{\partial H}{\partial x} \right\rangle$ and $- \frac{\partial H}{\partial \langle x \rangle}$ due to range of wave function.

Extended wave packets might force the particle away from it's average place of measurement.

$$\left\langle - \frac{\partial H}{\partial x} \right\rangle \underset{=}{-} \frac{\partial H}{\partial \langle x \rangle}$$

holds approximately for V general as long as Ψ is localized and holds exactly only for V that is a polynomial of maximally 2nd degree in X.