Classical Mechanics (in about an hour)

There are two key concepts to cover in classical mechanics that will apply to our understanding of quantum mechanics. They are the *Lagrangian* $\mathcal{L}$ and the *Hamiltonion* $\mathcal{H}$.

We start off easy by looking at Newton's Law in 1 dimension: $F = m\ddot{x}$. We can write this in a more "cheeky" way as:

$$F = m\ddot{x} = \dot{p} = -\frac{\partial V(x)}{\partial x}$$

Here we now have force as the negative gradient of the potential energy. We now want to express this relationship using the kinetic energy, $T = \frac{m\dot{x}^2}{2}$. We begin by writing T as a function of $\dot{x}$ [1] and take the partial derivative with respect to $\dot{x}$.

$$T = f(\dot{x}) = \frac{m\dot{x}^2}{2}$$

$$p_x = m\dot{x} = \frac{\partial T}{\partial \dot{x}} \rightarrow \textbf{Generalized Momentum}$$

and take the derivative w.r.t. time to get the force:

$$\dot{p_x} = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}}$$

and now Newton's Law can be written as:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} + \frac{\partial V(x)}{\partial x} = 0$$

We can now introduce the *Lagrangian* as $\mathcal{L}(x, \dot{x}, t) = T - V$. Now our version of Newton's law looks like:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0$$

So what is the advantage of $\mathcal{L}$? Well, for one thing, $\mathcal{L}$ can be formalized in a much more general way than Newton's Laws. If we introduce the generalized coordinates: $q_i(t), \dot{q}_i = \frac{d}{dt}q_i(t)$,and t, we can then generalize the Lagrangian to $\mathcal{L}(q_i, \dot{q}_i, t)$. Now we can describe any system as long as we have an appropriate set of variables to describe it. Ordinarily, $\mathcal{L}(x, \dot{x}, t) = T - V$. This leads us to the Euler-Lagrange equation:

$$\forall i, \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Example: The Power of the Lagrangian

Take a spinning disc with angular velocity ω. Place a fly a distance $r_\perp$ from the center. What is the kinetic energy of the fly?

[1] In general we write the kinetic energy as a function of three variables, $T = f(x, \dot{x}, t)$

In Cartesian coordinates this would be a very tough problem to figure out. Using the $\mathcal{L}$ and the canonical coordinates $(r_\perp, \phi)$, though, will greatly simplify the problem. We'll start by looking at the kinetic energy and assume that the potential energy is 0 ($V = 0$).

$$T = \frac{mr^2}{2}(\omega + \dot{\phi})^2 + \frac{m}{2}\dot{r}^2$$

The 2 generalized momenta in this case are the linear momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$

and the angular momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2(\omega + \dot{\phi})$$

A quick note on momentum:

$$if\ \frac{\partial \mathcal{L}}{\partial q_i} = 0,\ then\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0\ \Rightarrow\ \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = const.$$

$$\therefore\ \dot{p} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ \textbf{is conserved}$$

We can see another advantage of the $\mathcal{L}$ is that it enables us to find conserved properties. Looking back at our fly on a disc, we see that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{d}{dt}m\dot{r} = \frac{\partial \mathcal{L}}{\partial r} = mr(\omega + \dot{\phi})^2$$

Where $mr(\omega + \dot{\phi})^2$ is the *centrifugal force*. Looking at the "momentum" (really, the angular momentum) belonging to ϕ

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt}mr^2(\omega + \dot{\phi}) = 2mr\dot{r}(\omega + \dot{\phi}) + mr^2\ddot{\phi} = 0$$

Here the term, $2mr\dot{r}(\omega + \dot{\phi})$, represents the *Coriolis force* and $mr^2\ddot{\phi}$ is a *torque* that the fly experiences as it moves out in r. This torque is in the -$\hat{\phi}$ direction.

Principle of Least Action

So why do we need $\mathcal{L}$? Well, it's a precursor to $\mathcal{H}$, the *Hamiltonion*, it's used in its own right for path integrals, and it follows the *Principle of Least Action.*

Let's begin by looking at boundary values and initial conditions. In order to solve a second order differential equation you need at least 2 initial/boundary conditions. These could be:

Where is it at $t = 0, q_i(0)$?

How fast is it moving at $t = 0, \dot{q}_i(0)$?

Alternatively, we can replace the 2nd boundary condition with a different one:

Where is it at a later time, $q_i(T)$?

These, or any equivalent set of conditions, will allow you to solve a second order differential equation like the Euler-Lagrange Equations. Newtonian physics is completely deterministic. With a given set of boundary/initial conditions $q_i(0), q_i(T)$ we can uniquely "fix" a path, $[q_i(t)]$ [2]. In reality there are an infinite number of paths between any two points, however only one path corresponds to physical reality. This path is an extremum of the action S and follows the *Principle of Least Action*. Here, the action for any given path $[q_i(t)]$ connecting $q_i(0)$ with $q_i(T)$ is defined as

$$S\left([q_i(t)]\right) = \int_0^T \mathcal{L}(q_i(t), \dot{q}_i(t), t)dt$$

so it is a "function of a set of functions" or a functional. For an *extremum* of this functional, we need to find a path $[q_i(t)]_{extr}$ such that any slight variations of this path leave the action unchanged: $\delta S|_{[q_i]_{extr}} = 0$ (where the δ refers to small variations of the path around the extremum). This is similar to a function $f(x)$ whose derivative, $f'(x_{extr}) = 0$, gives an extremum of the function, and values of $f(x)$ close to this extremum are the same: $\delta f|_{x\,extr} = 0$. The principle of least action states that of all possible paths between a fixed initial and final position, the one that corresponds to a (local) extremum of the action is the one the system will actually take. It can be shown (see Shankar or CM next semester) that this requirement is identical with Euler-Lagrange equations.

The Hamiltonian

Now that we know $\mathcal{L}$, what is the *Hamiltonian* $\mathcal{H}$?

$$\mathcal{H} = T + V$$

However, $\mathcal{H}$ is a function of the generalized momenta p_i, **not** the velocities $\dot{q}_i$, $\mathcal{H}(q_i, p_i, t)$. The general procedure for finding $\mathcal{H}$ is as follows:

1) Find your generalized coordinates and find the Lagrangian
2) Find the *canonical momenta* $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$
3) define the Hamiltonian

$$\mathcal{H}(q_i, p_i, t) = \sum_i \dot{q}_i p_i - \mathcal{L}(q_i, \dot{q}_i, t)$$

replace all $\dot{q}_i$ with $f(q_i, p_i)$. Going back to our fly

$$p_r = m\dot{r}$$

$$p_\phi = mr^2(\omega + \dot{\phi})$$

[2] a path is completely specified if I give $q_i(t)$ for all times $0 \leq t \leq T$.

$$\mathcal{H} = \dot{r}p_r + \dot{\phi}p_\phi - \mathcal{L}$$

$$= m\dot{r}^2 + mr^2\omega\dot{\phi} + mr^2\dot{\phi}^2 - \frac{m}{2}\dot{r}^2 - \frac{m}{2}r^2\omega^2 - mr^2\omega\dot{\phi} - \frac{m}{2}r^2\dot{\phi}^2$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{m}{2}r^2\omega^2$$

$$= \frac{p_r^2}{2m} + \frac{1}{2}mr^2\left(\frac{p_\phi}{mr^2} - \omega\right)^2 - \frac{1}{2}mr^2\omega^2$$

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - p_\phi\omega$$

$\mathcal{H}$ is a conserved quantity if it doesn't depend on time explicitly. Hamilton's equations of motion are:

$$\dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i}$$

Going back to our fly one last time

$$\dot{r} = \frac{p_r}{m} \quad \dot{p}_r = \frac{p_\phi^2}{2mr^3}$$

$$\dot{\phi} = \frac{p_\phi}{2mr^2} - \omega \quad \dot{p}_\phi = 0$$