0.1 Symmetries

Free particle Given the Hamiltonian of the free particle

$$H = \frac{\vec{p}^2}{2m}; \tag{1}$$

we can see since it has no explictly dependence on the coordinates q that the momentum is conserved. Indeed the equation of motion for $\vec{p}$ is:

$$\frac{d\vec{p}}{dt} = -\vec{\nabla}H = 0. \tag{2}$$

Cylindrical coordinates The earlier example for the Hamiltonian of a fly moving on a rotating disk, in cylindrical coordinates, is :

$$H = \frac{p_\phi^2}{2mr^2} + \frac{p_r^2}{2m} - \omega p_\phi. \tag{3}$$

We can observe that:

$$\frac{dp_\phi}{dt} = \frac{\partial H}{\partial \phi} = 0, \tag{4}$$

it implies that p_ϕ is conserved. We note that in this case, p_ϕ is not the ordinary momentum, but the angular momentum L_z around the z-axis. So, invariance under rotations around the z-axis (Hamiltonian doesn't depend on ϕ) implies conservation of angular momentum along that axis.

Energy If the Hamiltonian H is not explictly dependent on time we have:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0, \tag{5}$$

so the value of H is conserved. If H represents the total energy of the system (not necessarily always true), this means energy is conserved.

More general transformations What if we want to make a more general transformation of coordinates that leaves the Hamiltonian unchanged but is not directly related to a change of a single canonical coordinate q? Consider, as an example, a system of two particles that interact only through a potential $V(\vec{r}_1 - \vec{r}_2)$ that depends only on their relative positions (very important case). The Hamiltonian for this case is

$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(\vec{r}_1 - \vec{r}_2) \tag{6}$$

This Hamiltonian is clearly unchanged if we add a constant offset $\Delta\vec{R}$ to both $\vec{r}_1$ and $\vec{r}_2$. However, $\Delta\vec{R}$ is no canonical variable and therefore it is not clear a priori which momentum might be conserved because of this invariance. One way to solve this is by a change of variables. Introduce

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M};\ M = m_1 + m_2 \tag{7}$$
$$\vec{r} = \vec{r}_1 - \vec{r}_2;\ \vec{P} = \vec{p}_1 + \vec{p}_2$$
$$\vec{p} = \frac{m_2\vec{p}_1 - m_1\vec{p}_2}{M};\ \mu = \frac{m_1 m_2}{M}.$$

The Hamiltonian can now be written as

$$H = \frac{\vec{P}^{\,2}}{2M} + \frac{\vec{p}^{2}}{2\mu} + V(\vec{r}) \tag{8}$$

as the reader can easily prove by substitution and by checking the Hamilton equations of motion. In this form it is clear that every one of the three coordinates of $\vec{R}$ are cyclical (i.e., the Hamiltonian doesn't depend on them) and therefore all three components of $\vec{P}$ are conserved: in a two-particle system with only internal forces, total momentum is conserved. Note that this new form of H is very useful, since one can now separately solve for the 6 coordinates describing the center-of-mass motion – $\vec{P} = const., \vec{R} = \vec{R}_0 + \vec{P}/M \cdot t$ – and the remaining coordinates $\vec{p}, \vec{r}$ describe the motion of a *single* "pseudo particle" of mass μ. The motion of the original two particles can then be easily deduced by inverting the equations 7.

Active and Passive Transformations So far, we have looked at cases where the Hamiltonian is invariant under a change of position of one or several coordinates. This corresponds to an *active* transformation: I imagine that move a system instantaneously from a position q_i, p_i in phase space to a new position $\bar{q}_i.\bar{p}_i$. Plugging these new coordinates into the same Hamiltonian, I find $H(\bar{q}_i.\bar{p}_i) = H(q_i, p_i) + \Delta H$. If the Hamiltonian is unchanged under this transformation, $\Delta H = 0$, we call it a "symmetry operation".

Alternatively, we can keep the positions and momenta of the system the same, but transform from one set of variables to another (e.g., shift of coordinate system, or rotation, or going from cartesian to spherical coordinates, etc.). In general, we have new coordinates $\bar{q}_i.\bar{p}_i$ that express the *same* physical state of the system as the old ones, so that we can write $q_j = q_j(\bar{q}_i.\bar{p}_i), p_j = p_j(\bar{q}_i.\bar{p}_i)$. In general, we get than a new functional form for the dependence of the Hamiltonian on these new coordinates: $\bar{H}(\bar{q}_i.\bar{p}_i) = H(q_j(\bar{q}_i.\bar{p}_i), p_j(\bar{q}_i.\bar{p}_i)) = H(\bar{q}_i.\bar{p}_i) + \Delta H$. However, by comparison with the result above, it is clear that if the Hamiltonian is invariant under the *active* transformation $q_i, p_i \to \bar{q}_i.\bar{p}_i$, it will also be invariant under the corresponding *passive* one.

0.2 Poisson brackets

In the remainder, we want to discuss a specific class of coordinate transformations engendered by a **generator**. For this, we first introduce the concept of the Poisson brackets between two dynamical variables $\omega(q_i, p_i), \lambda(q_i, p_i)$, both of which are functions of the canonical coordinates and momenta:

$$\{\omega, \lambda\}_{q_i, p_i} = \sum_{i=1}^{N} \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \Lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \Lambda}{\partial q_i} \right).$$

Some important features of these: They are distributive, linear in both first and second entry, and **anti**commutative. There are several more interesting relationships one can prove, all of which (surprise!) are equivalent to analog relationships for commutators of operators!

The Poisson brackets between canonical coordinates and momenta gives:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}. \tag{9}$$

These relationships are of extreme importance! In fact, it can be shown that any transformation that keeps these relationships intact is a "canonical" one, in the sense that the new coordinates and momenta fulfill the Hamilton Equations of motion. A few examples: Simply exchanging all momenta with all coordinates and giving minus-signs to the new coordinates will give the same result for the Poisson brackets. Also, the parity operation (changing the sign of *all* coordinates and momenta) leads to the same commutators and is therefore canonical.

In general, if $g(q_i, p_i)$ is any dynamic variable, we can see that:

$$\frac{\partial g}{\partial q_i} = \{g, p_i\} \qquad \frac{\partial g}{\partial p_i} = -\{g, q_i\}, \tag{10}$$

Another very important relationship is

$$\begin{aligned} \frac{dg}{dt} &= \frac{\partial g}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial g}{\partial q_i} \dot{q}_i + \frac{\partial g}{\partial p_i} \dot{p}_i \right) \\ &= \frac{\partial g}{\partial t} + \{g, H\} \end{aligned} \tag{11}$$

We can now introduce the concept of a *infinitesimal* coordinate transformation *generated* by some dynamic function $g(q_i, p_i)$ (the "generator") which can be written in this way:

$$\left\{ \begin{array}{l} \overline{q}_i = q_i + \epsilon \{q_i, g\} = q_i + \epsilon \frac{\partial g}{\partial q_i} \\ \overline{p}_i = p_i + \epsilon \{p_i, g\} = p_i - \epsilon \frac{\partial g}{\partial p_i} \end{array} \right. \tag{12}$$

where ϵ is a small parameter.

The same equations describe the change of *any* kinematic variable $\omega(q_i, p_i)$ under this transformation:

$$\overline{\omega} = \omega(\overline{q}_i, \overline{p}_i) = \omega + \epsilon \{\omega, g\} \tag{13}$$

which follows directly from Eqs. 12 and the chain rule. In particular, the variation of the Hamiltonian is $\delta H = \epsilon \{H, g\}$. If the Hamiltonian is invariant under the transformation generated by g, this means that the Poisson bracket $\{H, g\} = 0$. However, turning it around also means that $\frac{dg}{dt} = 0$ (assuming g doesn't depend explicitly on time) in which case g is a conserved quantity.

We conclude with a few examples:

1. If we choose $g = p_k$ for the generator, we can see that the only variable changed by the transformation is $q_k \to q_k + \epsilon \{q_k, p_k\} = q_k + \epsilon$. In this case, we can immediately interpret the transformation as a translation and therefore the small parameter $\epsilon = \delta q_k$. All other kinematical variables get changed by $\delta\omega = \{\omega, p_k\} \delta q_k$. *Momenta are generators of infinitesimal translations!* If the Hamiltonian is unchanged under this transformation, then p_k is conserved.

2. If we choose $g = H$, i.e., the Hamiltonian, we can see that any variable is changed as $\omega \to \omega + \dot{\omega}\epsilon$ which, by definition, is the value of ω after the (infinitesimal) time ϵ has elapsed. So here $\epsilon = \delta t$. The Hamiltonian is therefore the generator of infinitesimal translations in time. If it doesn't depend on time, it is therefore conserved.

3. Similarly, one can show that each component of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is the generator of infinitesimal rotations around its axis. (This is left as an exercise for the reader). Therefore, if the Hamiltonian is unchanged under rotations, the angular momentum is conserved.