Graduate Quantum Mechanics - Problem Set 10

Problem 1)

Using the matrix elements of the operator $\mathbf{L}_x$ in the subspace for $l = 1$ (see Shankar p. 327-328, in particular the 3rd block in the matrix 12.5.23), show that the matrix for arbitrary rotations around the x-axis (in the basis consisting of eigenstates $|l=1,m\rangle$ (m = +1, 0, -1) of $\mathbf{L}_z$) is given by

$$D_{mm'}(\theta) = \exp\left(-i\theta\frac{\mathbf{L}_x}{\hbar}\right) = \begin{pmatrix} \frac{1}{2}\cos\theta + \frac{1}{2} & -\frac{i}{\sqrt{2}}\sin\theta & \frac{1}{2}\cos\theta - \frac{1}{2} \\ -\frac{i}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{i}{\sqrt{2}}\sin\theta \\ \frac{1}{2}\cos\theta - \frac{1}{2} & -\frac{i}{\sqrt{2}}\sin\theta & \frac{1}{2}\cos\theta + \frac{1}{2} \end{pmatrix}$$

Show that applying this matrix for the case θ=180° on the eigenfunction $|l=1,m=1\rangle$ of $\mathbf{L}_z$ gives the same result as rotating explicitly the function $Y_1^1(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$ by 180° around the x-axis.

Hint: Rotating by 180° around the x-axis is the same as replacing θ by 180°– θ and ϕ by – ϕ. Also, note that $Y_1^{-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}$.

Problem 2)

A particle in 3 dimensions is described by a wave function $\Psi(x,y,z)$. We can make a variable substitution to spherical variables $(x,y,z) \to (r, \theta, \varphi)$ by defining an alternative wave function in terms of the new variables which describes the exact same state:
$\Phi(r, \theta, \varphi) = \Psi(x, y, z) = \Psi(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$.
Using the definition of $\mathbf{L}_x$ in position (x,y,z) space, show that its representation in terms of these new variables is

$$\mathbf{L}_x = -\frac{\hbar}{i}\left(\sin\varphi\frac{\partial}{\partial\theta} + \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right).$$

Hint: Simply apply this form of $\mathbf{L}_x$ to $\Phi(r, \theta, \varphi)$ and show that you get the same answer as applying $\mathbf{L}_x$ in Cartesian coordinates to $\Psi(x,y,z)$.

Problem 3)

Using tbe result for $\mathbf{L}_x$ above and the similar result for $\mathbf{L}_y$ (see Exercise 12.5.8 p. 334 in Shankar – you don't have to prove that):

$$\mathbf{L}_y = \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$$

show that Eq. 12.5.27 is true, i.e.

$$\mathbf{L}_\pm = \pm\hbar e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

for at least one of the two (you can then assume the other one is correct, as well).

Problem 4)

Using the results from Problem 3 and the fact that
$\mathbf{L}_-\mathbf{L}_+ = \mathbf{L}_x^2 + \mathbf{L}_y^2 - \hbar\mathbf{L}_z$
calculate the form of the total squared angular momentum operator $\mathbf{L}^2 = \mathbf{L}_x^2 + \mathbf{L}_y^2 + \mathbf{L}_z^2$ in spherical coordinates and show that the result agrees with Shankar's Eq. 12.5.36, p. 335