Graduate Quantum Mechanics - Problem Set 3

Problem 1)

Consider the matrix $\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

i) Is it hermitian?

ii) Find its eigenvalues and eigenvectors

iii) Verify that $U^\dagger \Omega U$ is diagonal, U being the matrix formed by using each normalized eigenvector as one of its columns. (Show that U is unitary!)

iv) Calculate the matrix $\exp(i\Omega) = \sum_{n=0}^{\infty} \frac{1}{n!}(i\Omega)^n$ (where, for any matrix, $M^0 = \mathbf{1}$). Show that it is unitary.

Problem 2)

Show that $\delta(ax-b) = \frac{1}{|a|}\delta(x-\frac{b}{a})$ by evaluating $\int_{-\infty}^{\infty} f(x)\delta(ax-b)\,dx$ for an arbitrary function $f(x)$. Consider the two cases $a > 0$ and $a < 0$ separately.

Problem 3)

Consider the "Theta-funcion" $\theta(x-x') = \begin{cases} 1 \text{ if } x \geq x' \\ 0 \text{ else} \end{cases}$. Show that $\delta(x-x') = \frac{d\theta(x-x')}{dx}$ by multiplying both the l.h.s. and the r.h.s. with an arbitrary square-integrable function $f(x)$ and integrating over all x.

Problem 4)

A stone dropping from a height h = 5m follows a path $y(t) = h - ½\, gt^2$ (g = 10 m/s^2) until it hits the ground after exactly 1 second. Now consider a more general path $y(t) = h - ½\, gt^2 + a\sin(\pi t)$ with arbitrary constant a (and t measured in seconds). Obviously, this path has the same start and end points (at the same times) as the correct one. Show that the action (the integral over the proper Lagrangian for this situation along the second path) between these two endpoints has an extremum for the case $a = 0$.

[This is a very specific test of the general statement about the "principle of least action" in the lecture. You must do the math explicitly; don't just quote the general principle.]