Spherical Bessel Functions

Spherical Bessel functions, $j_\ell(x)$ and $n_\ell(x)$, are solutions to the differential equation

$$\frac{d^2 f_\ell}{dx^2} + \frac{2}{x}\frac{df_\ell}{dx} + \left[1 - \frac{\ell(\ell+1)}{x^2}\right] f_\ell = 0.$$

Also useful are the combinations $h_\ell^{(1)}(x) = j_\ell(x) + i n_\ell(x)$ and $h_\ell^{(2)}(x) = j_\ell(x) - i n_\ell(x) = [h_\ell^{(1)}(x)]^*$ and the modified spherical Bessel functions $i_\ell(x) = i^\ell j_\ell(ix)$ and $k_\ell(x) = -i^\ell h_\ell^{(1)}(ix)$. For $\ell = 0$, the solutions are

$$j_0(x) = \frac{\sin x}{x}, \qquad n_0(x) = -\frac{\cos x}{x}.$$

From the series solution, with the conventional normalization [see George Arfken, *Mathematical Methods for Physicists* (1985)], it can be shown that

$$f_{\ell-1} + f_{\ell+1} = \frac{2\ell+1}{x} f_\ell(x), \qquad \ell f_{\ell-1} - (\ell+1) f_{\ell+1} = (2\ell+1)\frac{df_\ell}{dx},$$

or

$$\frac{d}{dx}\left[x^{\ell+1} f_\ell(x)\right] = x^{\ell+1} f_{\ell-1}(x), \qquad \frac{d}{dx}\left[x^{-\ell} f_\ell(x)\right] = x^{-\ell} f_{\ell+1}(x),$$

where f_ℓ can be any of j_ℓ, n_l, $h_\ell^{(1)}$, $h_\ell^{(2)}$. These two recurrence relations in turn lead back to the differential equation. Induction on ℓ leads to the Rayleigh formulas,

$$j_\ell(x) = (-1)^\ell x^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell j_0(x), \qquad n_\ell(x) = (-1)^\ell x^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell n_0(x).$$

Applied for $\ell = 1$ and $\ell = 2$, these give

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x},$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x, \qquad n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x.$$

From the Rayleigh expressions it is easy to extract limiting behaviors: For $x \ll l$, the solutions behave as

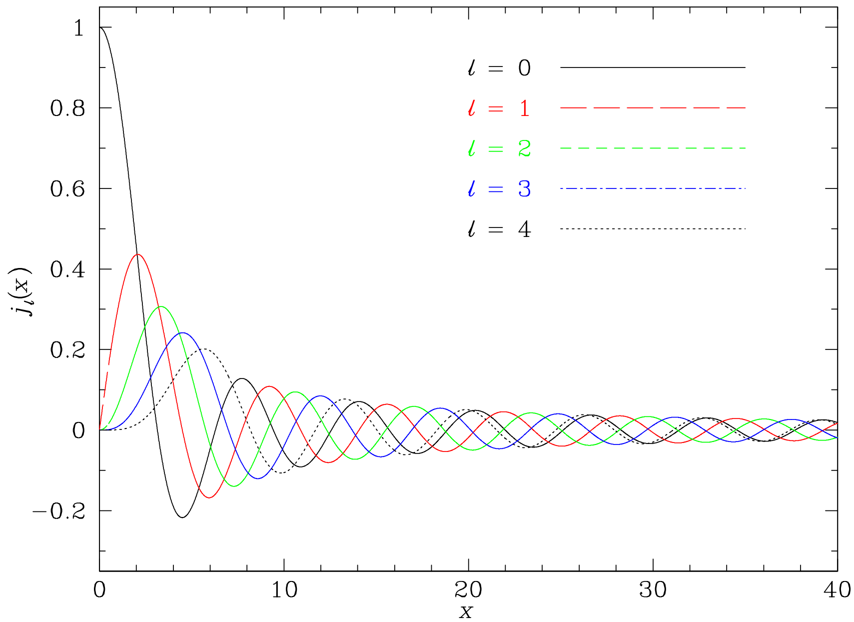
$$j_\ell \approx \frac{2^\ell \ell!}{(2\ell+1)!}\, x^\ell = \frac{x^\ell}{(2\ell+1)!!}, \qquad n_\ell \approx -\frac{(2\ell)!}{2^l \ell!}\, x^{-(\ell+1)} = -\frac{(2\ell-1)!!}{x^{\ell+1}}$$

and for $x \gg \ell$,

$$j_\ell \sim \frac{1}{x}\sin\!\left(x - \frac{\ell\pi}{2}\right), \qquad n_\ell \sim -\frac{1}{x}\cos\!\left(x - \frac{\ell\pi}{2}\right), \qquad h_\ell^{(1)} \sim (-i)^{\ell+1}\frac{e^{ix}}{x}.$$

Plots of j_0 through j_4 and n_0 through n_4 appear on the following page.

Spherical Bessel functions $j_\ell(x)$



Spherical Neumann functions $n_\ell(x)$